CHAPTER I

Introduction

1.1 Background

Jet engines, wakes behind airplanes and submarines, mixing layers, water disposal in rivers, chimney plumes and all kinds of motion in the atmosphere are a few examples of turbulent free shear flows which the engineers and the meteorologists as well, wish to predict. There are, in fact, many other flows of practical importance that need not to be boundary free as in the above flows. Examples of these flows are channel, pipe, and boundary layer flows. However, the process of free turbulent mixing is prominent in all these flows. Hence the theory of free shear flows, in general, applies to these flows as well.

The above classical flows have long been favorites for turbulence investigators because of the easy manner in which they can be generated in the laboratory. Another important characteristic of these flows, is their tendency to become fully developed and self-preserving (at least in principle) after a certain development region. This enables theoretical investigators to approximate the equations of motion based on physical grounds,such as order of magnitude analysis.

At the turn of the century the advances in the study of turbulent-flow problems were made primarily in the laboratory where basic insights into the general nature of turbulent flows were developed and the behavior of selected families of flows were studied systematically. For engineers and meteorologists there were only a limited number of useful tools, such as boundary layer prediction methods which solve the momentum integral equation with a high empirical content. Turbulent flow features such as sudden changes in boundary conditions, separation or recirculation could not be predicted by these early methods with any degree of reliability. Hence empirical work remained an essential ingredient in many engineering analysis.

Halfway into this century the computer began to have a major impact on engineering computations and the development of a theoretical model capable of predicting turbulent flows with a fair degree of accuracy began to attract many researchers in this field.

The exact equations that govern turbulent flows are well known; they are the Navier-Stokes equations. These equations, which are accepted as the fundamental basis for turbulent flow problems, are non-linear and strongly coupled; hence, an analytical approach leading to closed form solutions is not possible. Procedures exist to solve these equations numerically. However, the energy-dissipating eddies are so small that the computational mesh required must be so fine that realistic calculations cannot be carried out with present day computer hardware. Therefore it is customary to consider statistical properties of turbulence, which is often sufficient in providing engineers with the required information. This approach, however, leads to an infinite number of correlation equations that govern the turbulence properties.

A practical way to close the system of equations is to employ a turbulent model which approximates higher order correlations (moments) in terms of lower order moments that can be calculated. This approximation relies heavily on experimental data to determine the model empirical constants and functions. Therefore a reliable set of experimental data must be provided to serve as a basis for any theoretical prediction methods.

1.2 Theoretical Model

The turbulence models are classified either according to the number of partial differential equations they employ for turbulent quantities or by the order of the moment for which the transport equations are written.

The first turbulence model which has been applied to turbulent free shear flows with some success, is Prandtl's (1925) mixing-length hypothesis. This simple model relates the turbulent shear stress uniquely to the local mean velocity gradient. Then the partial differential equation for the mean flow is transformed to ordinary differential equations for which an analytical solution can be obtained. (see i.e. Appendix B) This model, among others of its class, often breaks down in many situations when there is more than one mechanism present, producing, in general, more than one length or velocity scale.

A second order model is expected to work better in most situations because it carries transport equations for second order quantities, so that many of the mechanisms responsible for the production of those quantities are represented accurately. Kolmogorov (1942), Prandtl (1945), Chou (1945) and Rotta (1951) laid the foundation for second order models of turbulence. However, analytical solutions for the resulting system of equations could not be obtained and a numerical one was not possible at that time.

By the early 70's when advances in computers and numerical methods overcame the mathematical difficulties, several predictions of turbulent free shear flows had been made with a fair degree of accuracy. Among the reported models are the (k-ε) model proposed by Jones and Launder (1972), (k-$k\ell$) model by Rodi and Spalding (1971) and the (k-ω) model by Spalding (1972). However, these prediction methods use model constants which were thought to be universal, but the calculations showed that they are not. For example, a set of constants that predict the flow for plane jets will not do so for the round jet.

Furthermore the two equation model used the eddy viscosity concept (e.g. $\nu_t \sim k^2/\varepsilon$),hence they do not keep track of the dynamics of all the second order correlations of importance. This led to the idea (Donaldson, 1971; Hanjalic and Launder, 1972b; Bradshaw, 1972) that the most promising class of turbulence models for making numerical calculations of such complex flows is that based on the solution of the approximated equations for the Reynolds stresses $\overline{u_i u_j}$ and indeed several proposals have been made (see section 2.4).

1.3 Scope and Object

In the past decade considerable success (Lumley and Khajeh-Nouri, 1974; Launder, Reece and Rodi, 1975; Reynolds, 1976 and Hanjalic and Launder 1976) have been made in predicting shear layer, jet wakes, channel flows, and boundary layers with reasonable degree of accuracy. There were, however, some unexplained differences between calculated and measured turbulent quantities.

These discrepancies arise from the neglect of some correlation terms in the governing equations, incomplete or inappropriate closure formulations for other correlations or simply not having the optimum values for the coefficients in closure formulations which may be functionally correct. For instance a set of constants in the closure formulations that gives good results for one flow situation sometimes does not work well for another flow. This is the case with the predictions for the two-dimensional and round jet flows (Launder and Morse, 1979).

Although some fundamental guiding principles, i.e. invarient modeling, have been used in formulating closure relations, much is developed by *ad hoc* assumptions. With appropriately adjusted constants some of these *ad hoc* closures have performed admirably well. However, one would like to develop closure formulations from first principles using rational procedures. Also it would be highly desirable that the model parameters and constants be determined as part of the calculation, or at least, determined from certain "key" basic experiments. Furthermore, closure formulations and the resulting theory should not violate certain mathematical or physical principles, e.g. conservation of mass and momentum.

Using a rational approach, Lumley (1978) formulated a second order model that is an orderly expansion about a homogeneous, stationary turbulence, the large scales of which have a Gaussian distribution. In this formulation care is taken to satisfy realizability conditions. This condition implies that non-negative quantities are never negative and Schwarz's inequality is satisfied. The key coefficients in this closure relation are functions of the local turbulent Reynolds number and anisotropy.

The primary aim of this dissertation is to consider the above closure formulation and investigate the functional form of the model parameters based on the available data for a homogeneous decaying axisymmetric turbulent flow. The closed Reynolds stress and dissipation equations are transformed to curvilinear coordinates for the use in the axisymmetric jet calculations.

The similarity forms of the resulting system of equations for plane and axisymmetric flow are solved numerically to determine the equilibrium behavior of turbulent (isothermal) fully developed and self-similar jets. The results are compared with available experimental data with the emphasis on conservation of momentum.

C. B. Baker (1980) raised the question about the validity of the axisymmetric jet measurements, since they failed to conserve momentum. He analyzed the data of Wygnanski and Fiedler (1969) for an axisymmetric self-preserving jet and argued that the measured mean velocity profile conserves only half of the momentum added at the source. (See also George, Seif and Baker, 1981).

On the other hand the most recently measured and calculated profiles are fairly in good agreement with Wygnanski and Fiedler profiles when they are normalized with their respective centerline value of the mean velocity. Hence part of this study (chapter 5) is devoted to examination of the jet data (plane and axisymmetric) in contrast with the results of theoretical predictions.

CHAPTER 2

The Reynolds Stress Closure

2.1 Equations for the Mean Flow

The equations that govern the mean motion of an incompressible isothermal turbulent flow are obtained from the Navier-Stokes equations. By decomposing the instantaneous velocity and pressure into a mean and turbulent component and by taking the time average of all terms, the following equations will result (see Tennekes and Lumley 1972).

Conservation of mass:

$$U_{i,i} = 0 \tag{2.1}$$

Conservation of momentum:

$$\rho\dot{U}_i + \rho U_j U_{i,j} = -P_{,i} + (\mu U_{i,j} - \rho\overline{u_i u_j})_{,j} \tag{2.2}$$

where the overbar denotes the time average, the overdot stands for the partial derivative with respect to time,and the subscripts after the commas denote the partial differentiation, eg. $U_{i,j} = \frac{\partial U_i}{\partial x_j}$. The new unknown $\rho\overline{u_i u_j}$ in the momentum equation is the contribution of the turbulent motion to the mean stress tensor. It is known as the Reynolds stress in honor of Reynolds who first developed equation (2.2) in (1785). The Reynolds stress $\rho\overline{u_i u_j}$ has nine components and hence introduces nine unknowns to the equation of motion; however,since it is a symmetric tensor ($\overline{u_i u_j} = \overline{u_j u_i}$) the number of unknowns is reduced to six, three normal and three tangential components.

The aim of any prediction method in turbulent modeling is to solve the momentum equation for U_i,but because of the presence of $\overline{u_i u_j}$ in the momentum equation, the system of equations in (2.1) and (2.2) do not constitute a closed set. Closing this set of equations has been of major concern for over a century. An earlier closure, which is known today as the zero order model, was originally proposed by Boussinesq in 1877. This simple closure model assumes that the shear stress is proportional to the mean velocity gradient. This approximation predicts the velocity and shear stress profiles for the self preserving turbulent jet (see Appendix C) with a good degree of accuracy over most of the flow region, but it fails to do so when the turbulence is in non-self-preserving state. However, with the advances of electronic facilities researchers have tried to develop a universal method to predict the Reynolds stress accurately. The most direct way to determine $\overline{u_i u_j}$, of course, is to solve a transport equation for all non-zero components of the Reynolds stress. Such an equation, in fact, does exist and it will be discussed in the following section.

2.2 The Reynolds Stress Equation

A transport equation that governs the Reynolds stress can be derived in the following way. The equation for the component i of the instantaneous velocity (U_i+u_i) is multiplied by u_j and the equation for the j component $U_j+u_j)$ is multiplied by u_i. Summing of the two equations and taking the time average yields the desired equation for $\overline{u_i u_j}$ (see Hinze,1959):

$$\dot{\overline{u_i u_j}} + U_k(\overline{u_i u_j})_{,k} = - (\overline{u_i u_k}\, U_{j,k} + \overline{u_j u_k}\, U_{i,k}) + \overline{\frac{p}{\rho}\,(u_{j,i} + u_{i,j})}$$

(i) = convection (ii) = production (iii) = pressure strain

$$- [\overline{u_i u_j u_k} - \nu(\overline{u_i u_j})_{,k} + \overline{\frac{p}{\rho}\,(u_i \delta_{jk} + u_j \delta_{ik})}]_{,k}$$

(iv) = diffusion

$$- 2\,\nu\,\overline{u_{i,k} u_{j,k}} \qquad (2.3)$$

(v) = dissipation

As it can be seen from equation (2.3), further unknowns, such as the triple correlation and pressure velocity correlations, have been introduced. This, of course, adds to the complexity of the situation. Transport equations for the third order statistical moment $\overline{u_i u_j u_k}$ can be again derived in a way similar to the above; however, the number of unknowns will grow faster than the number of equations. Closing the system in equation (2.3) at the second order level (the Reynolds stress closure) will be discussed later in this chapter.

2.3 The Kinetic Energy Equation

For future reference let us take a look at the turbulent kinetic energy equation. Contraction of equation (2.3) leads to an important equation, the kinetic energy equation of the turbulence:

$$\dot{\overline{q^2}} + U_j \overline{q^2}_{,j} = -[\overline{u_j(q^2 + 2\tfrac{p}{\rho})}]_{,j} - 2\, \overline{u_i u_j}\, U_{i,j} + 2\, \nu(\overline{u_i u_{i,j}})_{,j}$$
$$\text{(i)} \qquad\qquad \text{(ii)} \qquad\qquad \text{(iii)} \qquad\qquad \text{(iv)}$$

$$- 2\nu\, \overline{u_{i,j} u_{i,j}} \qquad (2.4)$$
$$\text{(v)}$$

where $\overline{q^2} = \overline{u_i u_i}$.

Equation (2.4) states: The change in (i), the turbulent kinetic energy per unit mass and unit time including the convection transport by the mean motion, is equal to (ii), the convective diffusion by the turbulence of the total mechanical energy or the work by the total dynamic pressure of the turbulence, plus (iii), the work of deformation of the mean motion by the turbulence stresses, plus (iv), the work done by the viscous stresses of the turbulent motion, plus (v), the viscous dissipation by turbulent motion. To close the system of equations (2.2) and (2.4) at this level, which is known in the literature as the one equation model, the terms on the right hand side of equation (2.4) will be approximated employing the eddy viscosity concept and specifying a characteristic length scale (see Reynolds 1976). However, if we go one step further and derive an additional transport equation for the dissipation ε, then we have the so called two equation model. This model eliminates the need for specifying a characteristic length scale as function of position throughout the flow by defining the eddy viscosity as $\nu_t \sim \overline{q^2}^2/\varepsilon$. A detailed discussion of this model and its application will be presented in Chapter three.

2.4 The Dissipation Rate Equation

An exact transport equation for the dissipation rate of turbulent kinetic energy (i.e., the correlation $\nu\,\overline{u_{i,\ell}u_{i,\ell}}$) can be developed from the Navier-Stokes equations for the fluctuating velocities by appropriate differentiation, multiplication and averaging. The resulting equation can be written as (see Daly & Harlow 1970):

$$\dot{\varepsilon} + U_k\varepsilon_{,k} = \underset{\text{(i)}}{-2\,\nu\,\overline{u_{i,k}u_{i,j}u_{k,j}}} - \underset{\text{(ii)}}{2\,\overline{(\nu u_{i,jk})^2}}$$

$$\underset{\text{(iii)}}{-(\overline{u_k\varepsilon} + 2\,\frac{\nu}{\rho}\,\overline{u_{k,i}P_{,j}} - \nu\varepsilon_{,k})_{,k}}$$

$$\underset{\text{(iv)}}{-2\,\nu(\overline{u_{i,j}u_{k,j}} + \overline{u_{j,i}u_{j,k}})\,U_{i,k}}$$

$$\underset{\text{(v)}}{-2\,\nu\overline{u_k u_{i,j}}\,U_{i,jk}} \tag{2.5}$$

It is an extremely difficult task to consider equation (2.5) in its entirety. Luckily for high Reynolds numbers flow (i.e. most of the turbulent flows) a great simplification will result when an order of magnitude analysis is employed (Tennekes and Lumley 1972). The terms (i) and (ii) which represent the generation of ε by the stretching of vortex filaments, and the destruction through the tendency of viscosity to reduce the instantaneous velocity gradients are the most dominant terms. However their difference, which really matters, is nearly the same order of magnitude as the transport terms (iii). The terms (iv) and (v) are smaller than other terms by at least a factor of $R_\ell^{1/2}$, where R_ℓ is the turbulent Reynolds number; therefore, these terms can be safely ignored. Hence equation (2.5) can be written as

$$\dot{\varepsilon} + U_k \varepsilon_{,k} = - \underset{\text{(i)}}{(\overline{u_k \varepsilon} + 2 \frac{\nu}{\rho} \overline{P_{,i} u_{k,i}})_{,k}}$$

$$- \underset{\text{(ii)}}{2 \nu \overline{u_{i,k} u_{i,j} u_{k,j}}} - \underset{\text{(iii)}}{2 \overline{(\nu u_{i,jk})^2}} \qquad (2.6)$$

Still the terms on the right hand side of equation (2.6) add further unknowns into the equation set governing the Reynolds stress. These terms are not directly accessible to measurement and therefore their approximation can be only verified indirectly by determining whether the predicted distribution of ε is consistent with the measured variation of the turbulent kinetic energy through a particular shear flow. Modeling of the transport terms, (i), and production-destruction terms, (ii) and (iii), in equation (2.6) will be included in the analysis of the Reynolds stress closure.

2.5 The Reynolds Stress Closure Approximation

The Reynolds stress model begins with the equations (2.1), (2.2), (2.3) and (2.6). In order to solve equation (2.3) for $\overline{u_i u_j}$, some information about the higher order moments $\overline{u_i u_j u_k}$ and $\overline{P u_{i,j}}$ must be provided. Those terms will be approximated as functions of the lower order moments. Such approximations will rely heavily on experimental data to determine the proportionality constants and certain key parameters.

The idea of proposing a model like (2.3) was first suggested by Rotta (1951). Some predictions have been recently made following this idea by Daly and Harlow (1970), Reynolds (1970) Donaldson (1971), Noat, Shavit and Wolfstein (1972), Hanjalic and Launder (1972) and Lumley and Khajah-Nouri (1974), to name a few. However there have been widely different views on how to treat the third order moments, the triple velocity correlation in particular. Before we proceed with the analysis of closing the Reynolds stress equations, a new arrangement of the terms involved in equation (2.3) will be made. For convenience in later analysis we will separate the effects of the various terms to be modeled and group them according to their rules and functions in the equations of motion.

An expression for the pressure can be obtained by taking the divergence of the Navier-Stokes equations for the fluctuating velocity component u_i. The result is (Lumley, 1978)

$$-\frac{p_{,ii}}{\rho} = 2\, U_{i,j}u_{j,i} + \overline{u_{i,j}u_{j,i}} - \overline{u_i u_{j,ij}} \tag{2.7}$$

The right hand side of equation (2.7) contains two types of terms. The first term is linear in the fluctuating velocity and related to the mean velocity gradients while the second and third are nonlinear in the fluctuating velocity. If we conveniently split the pressure such that

$$-\frac{p^{(1)}_{,ii}}{\rho} = 2\, U_{i,j}u_{j,i} \tag{2.8}$$

$$-\frac{p^{(2)}_{,ii}}{\rho} = \overline{u_{i,j}u_{j,i}} - \overline{u_i u_{j,ij}} \tag{2.9}$$

where the correlations with $p^{(1)}$ and its gradients are known as the "rapid terms". While the correlations with $p^{(2)}$ are known as the "return-to-isotropy"terms. This part is responsible for the return of anisotropic turbulence to isotropy in the absence of other disturbing effects. Based on the separation of the pressure we may write the pressure strain term in equation (2.3) as follows:

$$\overline{\frac{p}{\rho}(u_{j,i}+u_{i,j})} = \overline{\frac{p^{(1)}}{\rho}(u_{i,j}+u_{j,i})} + \overline{\frac{p^{(2)}}{\rho}(u_{i,j}+u_{j,i})} \qquad (2.10)$$

where $p^{(1)}$ and $p^{(2)}$ are given by equations (2.8) and (2.9).

The primary function of the viscous terms($2\nu\,\overline{u_{i,k}u_{j,k}}$)is to dissipate $\overline{u_i u_j}$ into heat; however, it can also cause interchange of energy among the components of $\overline{u_i u_j}$. If we add and subtract the trace of the viscous term which is twice ε, the dissipation rate of the turbulent energy, we get (Lumley 1978)

$$2\nu\,\overline{u_{i,k}u_{j,k}} = (2\,\overline{\nu u_{i,k}u_{j,k}} - \frac{2}{3}\varepsilon\delta_{ij}) + \frac{2}{3}\varepsilon\delta_{ij} \qquad (2.11)$$

where now the terms in the bracket act to interchange energy amoug the components of $\overline{u_i u_j}$ while $\frac{2}{3}\varepsilon\delta_{ij}$ is responsible for their dissipation. Substituting equations (2.10) and (2.11) into (2.3),the Reynolds stress equation takes the form

$$\begin{aligned}
&\underset{\text{(i)}}{\dot{\overline{u_i u_j}} + U_k \overline{u_i u_{j,k}}} = -\underset{\text{(ii)}}{(\overline{u_i u_k}\,U_{j,k} + \overline{u_j u_k}\,U_{i,k})} \\
&\quad - \underset{\text{(iii)}}{[\overline{u_i u_j u_k} - \overline{\nu(u_i u_j)}_{,k} + \overline{\frac{p}{\rho}(u_i\delta_{jk} + u_j\delta_{ik})}]_{,k}} \\
&\quad + \underset{\text{(iv)}}{[\overline{\frac{p^{(2)}}{\rho}(u_{i,j} + u_{j,i})} - 2\,\overline{\nu u_{i,k}u_{j,k}} + \frac{2}{3}\varepsilon\delta_{ij}]} \\
&\quad + \underset{\text{(v)}}{\overline{\frac{p^{(1)}}{\rho}(u_{i,j}+u_{j,i})}} - \underset{\text{(vi)}}{\frac{2}{3}\varepsilon\delta_{ij}}
\end{aligned} \qquad (2.12)$$

Equation (2.12) states that the substantial change of $\overline{u_i u_j}$ during convection,(i),is equal to (ii), the production of the Reynolds stress from the mean flow, plus (iii),the gradient of the transport from the velocity and pressure fluctuation, plus (iv),the return to isotropy of the Reynolds stress, plus (v),the rapid straining change of $\overline{u_i u_j}$ plus (vi), the mechanical dissipation of $\overline{u_i u_j}$ into heat.

2.6 A Model for the Dissipation Equation

Several forms analogous to that given by equation (2.6) have been proposed to model the dissipation equation. Hanjalic and Launder (1972) suggested the following transport equation for ε:

$$\dot{\varepsilon} + U_k \varepsilon_{,k} = C_\varepsilon\left(\frac{\overline{q^2}}{\varepsilon} u_k u_\ell \varepsilon_{,\ell}\right)_{,k} - C_{\varepsilon 1} \varepsilon \frac{u_i u_k}{\overline{q^2}} U_{i,k} - C_{\varepsilon 2} \frac{\varepsilon^2}{\overline{q^2}} \qquad (2.13)$$

(i) = diffusion (ii) = production (iii) = destruction

where C_ε, $C_{\varepsilon 1}$ and $C_{\varepsilon 2}$ are model constants to be determined from experimental data. Lumley and Khajah-Nouri (1974) proposed an equation of the form,

$$\dot{\varepsilon} + U_k \varepsilon_{,k} = - (\varepsilon u_k)_{,k} + \psi \frac{\varepsilon^2}{\overline{q^2}} \qquad (2.14)$$

where the dimensionless invariant function ψ contains the entire mechanism for production and destruction of ε. Determining the functional dependence of ψ is not an easy task; however,since ψ represents the production/destruction of ε it is reasonable to assume that ψ depends on the Reynolds stress, the mean velocity gradients,and the dissipation. In fact, there are a large number of invariants that can be formed on which the function ψ might depend. This list includes:

$$\psi = [II, III, b_{ij}(\overline{q^2}/\varepsilon U_{i,j}), b_{ij}^2 (\overline{q^2}/\varepsilon U_{i,j}) + \ldots, R_\ell] \qquad (2.15)$$

where

$$II = -b_{ij}b_{ij}/2 \tag{2.16}$$

$$III = b_{ij}b_{jk}b_{ki}/3 \tag{2.17}$$

and b_{ij} is the anisotropy tensor which is defined by

$$b_{ij} = \frac{\overline{u_i u_j}}{\overline{q^2}} - \frac{1}{3}\delta_{ij} \tag{2.18}$$

Note that b_{ij} is symmetric, dimensionless, has zero trace, and vanishes identically when the turbulence is isotropic.

Finally R_ℓ is the turbulence Reynolds number defined to be:

$$R_\ell = \overline{q^2}^{\,2}/(9\varepsilon\nu) \tag{2.19}$$

The factor of 9 is included in the definition so that R_ℓ reduces to the traditional form $R_\ell = u\ell/\nu$, since $\varepsilon \sim u^3/\ell$ and $\overline{q^2} \equiv 3\,u^3$.

If we assume that the velocity gradients are not too large, then we can expand ψ as given by equation (2.15) into a power series in b_{ij}:

$$\psi = \psi_o + \psi_1\, b_{ij}\, \frac{\overline{q^2}}{\varepsilon} U_{i,j} + \psi_2\, b_{ij}^2\, \frac{\overline{q^2}}{\varepsilon} U_{i,j} + \ldots \tag{2.20}$$

where the coefficients are functions of II, III and the Reynolds number. If we substitute equation (2.20) into equation (2.14) and retain only first order terms, the equation for ε becomes:

$$\dot{\varepsilon} + U_k\varepsilon_{,k} = -(\varepsilon u_k)_{,k} - \psi_1\, \varepsilon\, b_{ij}\, U_{i,j} - \psi_o \frac{\varepsilon^2}{\overline{q^2}} \tag{2.21}$$

Now what remains is to model the transport term of ε and to determine the functional dependence of ψ_o and the value of ψ_1. However, there is no reliable comprehensive set of data for a general flow from which the various parameters in the closure can be evaluated at the same time. The best that can be done is to assume a hierarchy of simpler flows in which progressively more (or less) of the unknown terms vanish identically sothat the unknown coefficients which occur in the assumed closure for the remaining terms can be progressively evaluated. This progressive evaluation can be utilized since the closure parameters are assumed to be universal.

2.7 Decay of Isotropic Turbulence

To determine ψ_0 let us consider a simple, homogeneous, decaying, isotropic turbulence without the mean strain. For this class of flows the governing equations are the turbulent kinetic energy equation (2.4) and the dissipation equation (2.21). They simplify to:

$$\dot{\overline{q^2}} = -2\varepsilon \tag{2.22}$$

$$\dot{\varepsilon} = -\psi_0 \frac{\varepsilon^2}{\overline{q^2}} \tag{2.23}$$

If we assume that the turbulent energy decays as

$$\overline{q^2} \sim t^{-n} \tag{2.24}$$

where t denotes time and n is an exponent to be determined, then from equation (2.22) and (2.23) we obtain the following relation for ψ_0:

$$\psi_0 = \frac{2(n+1)}{n} \tag{2.25}$$

The asymptotic values for the exponent n according to Hinze (1975) is that n=1 in the limit $R_\ell \to \infty$ and n=2.5 when $R_\ell \to o$. This suggests that:

$$2.8 \leq \psi_0 \leq 4.0$$

The grid-generated isotropic turbulence experiments of Comte-Bellot and Corrsin (1966) indicate that the exponent n lies between 1.2 and 1.35. For a constant ψ_0 model, various numerical values for ψ_0 has been proposed. Hanjalic and Launder used ψ_0 = 4.0; Launder, Reece and Rodi (1975) used ψ_0 = 3.8; Zeman & Lumley (1976) suggested that ψ_0 = 3.8; and Reynolds (1976) recommended that $\psi_0 = 11/3$.

A constant ψ_0 model had both successes and failures in the 70's. More discussion on these models will be presented in Chapter 3 where the two equations for the k-ε model are reviewed. More recently Lumley (1974), Hanjalic and Launder (1976), Reynolds (1976) and Lumley and Newman (1977), and others have proposed a variable ψ_0. Hanjalic and Launder (1976) assumed that ψ_0 depends on the turbulent Reynolds number only and based their formulation of ψ_0 on the data of Batchelor and Townsend (1948) which is, in fact, a low Reynolds number flow. They suggested;

$$\psi_0 = 3.6\ [1 - .23\ \exp\ (-.14\ R_\ell^2)] \qquad (2.26)$$

Lumley and Newman (1977) have also included in the dependence of ψ_0 the first invariant of the anisotropy II. The form they proposed based is on the experimental data of Comte-Bellot and Corrsin (1966) for which $R_\ell > 100$; they suggested:

$$\psi_0 = \frac{14}{5} + .98\ \exp\ (-2.83/\sqrt{R_\ell})\ [1 - \ell n\ (1 - 55\ II)] \qquad (2.27)$$

Chung(1978) reexamined Batchelor and Townsend (1948) and Comte-Bellot and Corrsin (1966) data and suggested the following:

$$\psi_0 = 3.55\ [1 - .211\ \exp(-R_\ell^2/2.5)] + .45\ \exp\ (-\ 36.7/\sqrt{R_\ell}) \qquad (2.28)$$

He argued that the inclusion of II in the ψ_0 dependence is correct based on theoretical grounds, but unnecessary in practice. Figure 2.1 shows a comparison of the above three models for ψ_0 for a wide range of R_ℓ. As can be seen from Figure 2.1, the Lumley and Newman (1977) and the Chung (1978) formulations for ψ_0 approach nearly the same value when $R_\ell > 200$. From Figure 2.1 it can be seen that the formulation of Lumley and Newman (equation (2.27)) is in error for small Reynolds numbers; in particular, when

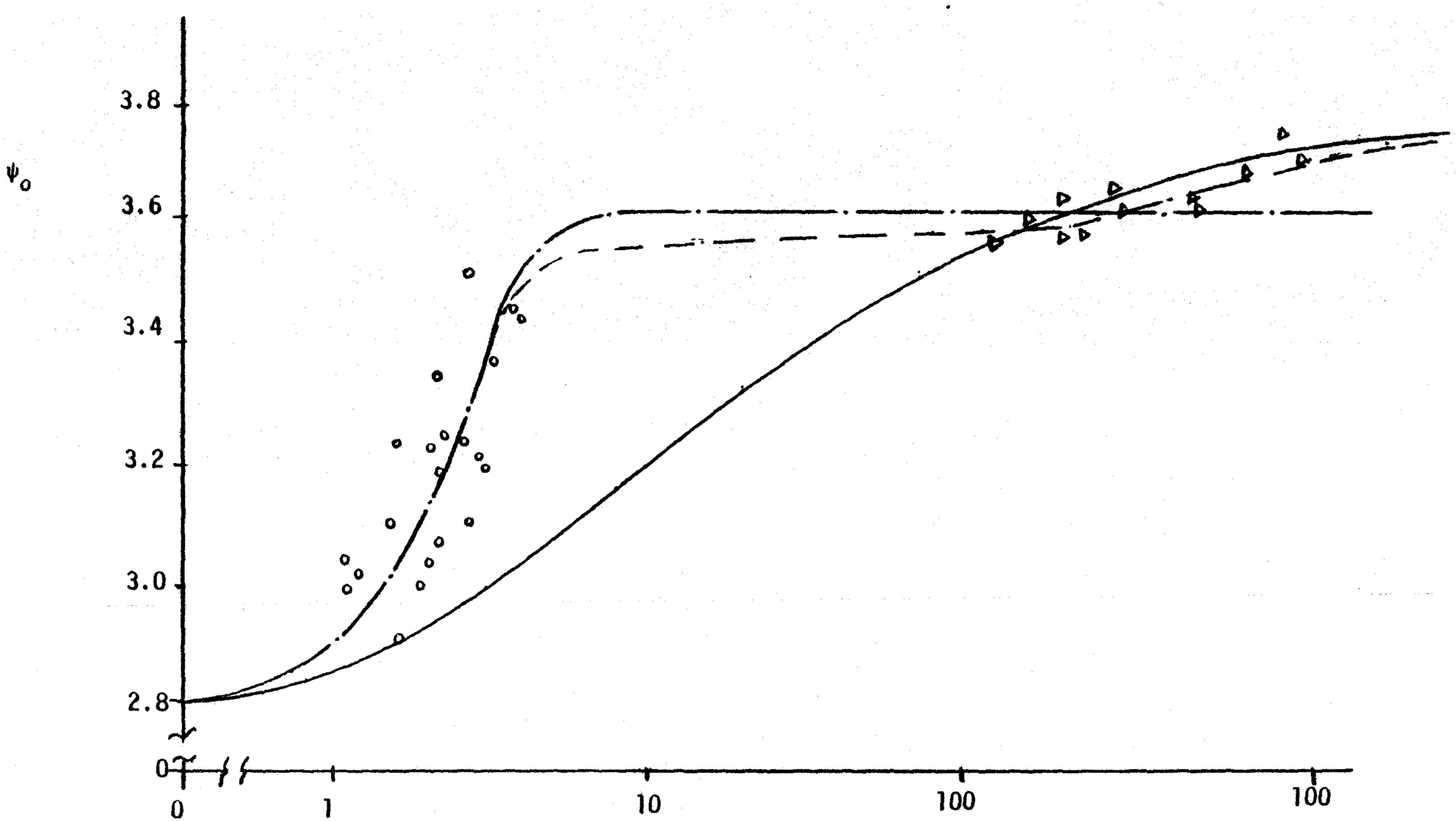

Figure 2.1 Rate of Energy Decay

(——) Lumley and Newman (-·-), Hanjalic and Launder (---), Chung,
data of: (o) Batchelor and Townsend (1942), (Δ) Comte-Bellot and Corrsin (1966)

$$5 < R_\ell < 100$$

This is because the formulation of equation (2.27) was based on high Reynolds number free shear flows. More important is that the basic assumptions in forming the Reynolds stress and the dissipation equations are valid only for high Reynolds number flows. Equation (2.27) worked remarkably well in the wake calculations of Taulbee and Lumley (1980) and in the present study; for both flows the turbulent Reynolds number is in the high range, $R_\ell > 400$.

2.8 Determination of ψ_1

In order to determine ψ_1 let us consider a turbulent flow subject to constant strain. Examples of such flows are:

a) Turbulence distorted by plain strain (Tucker and Reynolds 1968).

b) Turbulence passing through contractions (Uberoi 1956).

c) Nearly homogeneous shear flows (Champagne et al. 1970).

The rate of strain was nearly constant in all of the above experiments. For these flows the energy and dissipation equations can be written as:

$$\frac{D\overline{q^2}}{Dt} = - \overline{u_i u_j}\, U_{i,j} - 2\varepsilon \qquad (2.29)$$

$$\frac{D\varepsilon}{Dt} = - \psi_1\, \varepsilon\, b_{i,j} - \psi_o \frac{\varepsilon^2}{\overline{q^2}} \qquad (2.30)$$

where $\frac{D}{Dt}$ stand for total derivative. The empirical parameter ψ_1 can be obtained from equation (2.30) using the above experimental data. In fact Rodi (1972) predicted the flows of Tucker and Reynolds (1968) and Champagne et al.(1970) quite well by taking $\psi_1 = 2.4$ and $\psi_1 = 4$ respectively, however when $\psi_1 = 2.4$ was used in other flows, it gave values of ε which were too low and there was hardly any decay of the kinetic energy. Reynolds (1976) reexamined the same experiments cited above and suggested that $\psi_1 = 2.0$ is suitable for most free shear flows. Launder et al. (1972) give $\psi_1 = 3.10$ and Hanjalic and Launder (1972) used $\psi_1 = 2.9$ in the calculations. The variation of ψ_1 from one flow to another suggests that ψ_1 can not be a universal constant, but some function of the state of turbulence. It is desirable to determine the functional dependence of ψ_1; however, lack of reliable experimental data makes it difficult at this time to predict a variable ψ_1. The value of $\psi_1 = 2.0$ seems to work well in the present study with the ψ_0 given by equation (2.28).

2.9 Return to Isotropy

In the absence of other influences the turbulence components interchange energy so as to become more nearly equal. This return to isotropy is produced mainly by the pressure fluctuation term and in part by the viscous term found in term [iv] of equation (2.12). The problem now is to analyze the correlations that control the return to isotropy. In order to do so let us consider a homogeneous flow without mean velocity gradients. The Reynolds stress equation (2.12) for this flow becomes:

$$\dot{\overline{u_i u_j}} = [\overline{\frac{p^{(2)}}{\rho}(u_{i,j} + u_{j,i})} - 2\,\nu \overline{u_{i,k} u_{j,k}} + \frac{2}{3}\,\varepsilon\delta_{ij}] - \frac{2}{3}\,\varepsilon\delta_{ij} \quad (2.31)$$

The quantity in the square bracket is responsible for the return to isotropy. It is a symmetric tensor, it has zero trace, and it vanishes identically when the turbulence is isotropic.

The first return to isotropy model was proposed by Rotta (1952). He assumed that the return to isotropy is linearly related to the deviation from isotropy; that is,

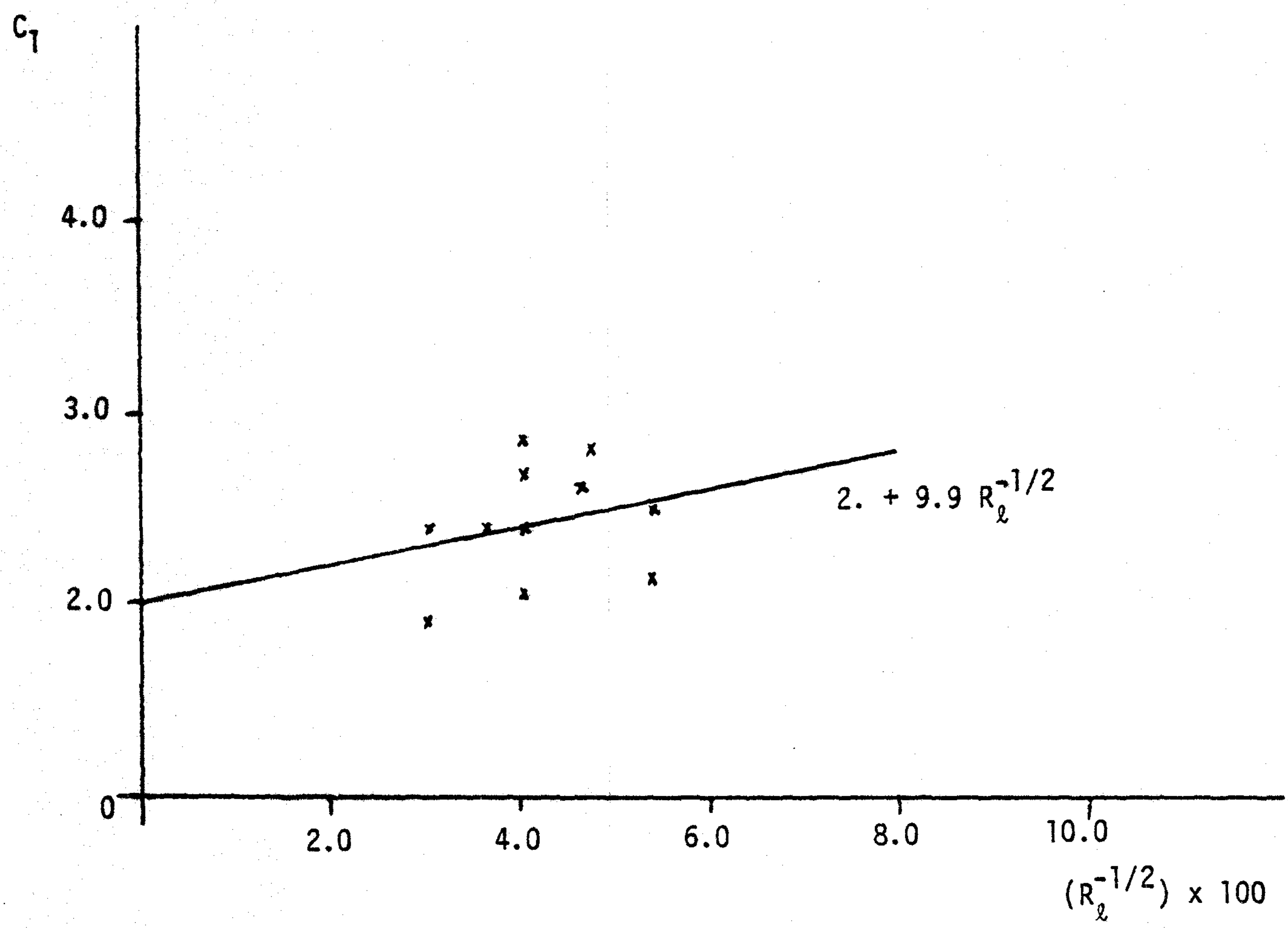

Figure 2.2 Variation of the Return to Isotropy Function with the Turbulent Reynolds number for a fixed value of (-II). Data of Comte-Bellot and Corrsin (1966).

$$\overline{\frac{p^{(2)}}{\rho}(u_{i,j} + u_{j,i})} = - C_1 \varepsilon\, b_{ij} \tag{2.32}$$

where C_1 is an empirical parameter to be determined and b_{ij} is an anisotropic tensor which satisfies the prescribed requirements and defined by equation (2.18). In previous works, C_1 has been taken as a universal constant with values ranging from $C_1 = 2.5$ as suggested by Reynolds (1976) to the value of $C_1 = 6.7$ as given by Wyngaard and Cote (1974). More recently Lumley and Khajeh-Nouri (1974), Hanjalic and Launder (1976), and Lumley and Newman (1977) have proven that C_1 may be a function of several variables. Now since we have included part of the viscous effect in the return to isotropy term we may write equation (2.32) as:

$$\overline{\frac{p^{(2)}}{\rho}(u_{i,j} + u_{j,i})} - 2\,\overline{\nu u_{i,k} u_{j,k}} + \frac{2}{3}\varepsilon\delta_{ij} = - \varepsilon\, \phi_{ij} \tag{2.33}$$

where the inclusion of ε in the right side of Equation (2.33) makes the expression dimensionally correct with the function ϕ_{ij} dimensionless. ϕ_{ij} has zero trace and is responsible for the return to isotropy. If we replace ϕ_{ij} by $C_1 b_{ij}$ as is commonly used in the literature, and substitute equation (2.33) into equation (2.31) we get,

$$\dot{\overline{u_i u_j}} = -C_1 b_{ij}\, \varepsilon - \frac{2}{3}\varepsilon\delta_{ij} \tag{2.34}$$

where C_1 is the same as in equation (2.32).

2.10 Determination of C_1

It can be seen from equation (2.34) that C_1 can be determined by the history of the Reynolds stress, the present values of b_{ij}, and the dissipation. Hence we can write in general:

$$C_1 = C_1\,(II, III, R_\ell) \tag{2.35}$$

Now if we consider equation (2.34) as a governing equation for the anisotropic tensor b_{ij} it can be shown after simple manipulation that

$$\frac{d}{dt}(b_{ij}) = -\frac{\varepsilon}{\overline{q^2}}(C_1-2)b_{ij} \tag{2.36}$$

If we define a non-dimensional time τ by $\overline{q^2}/\varepsilon$, then

$$\left(\frac{\varepsilon}{\overline{q^2}}\right)dt = d\tau \tag{2.37}$$

and the equation for b_{ij} can be written in the form

$$\frac{d}{d\tau}(b_{ij}) = -(C_1-2)\, b_{ij} \tag{2.38}$$

From equation (2.38) it can be seen that we must have

$$C_1 - 2 \geq 0 \tag{2.39}$$

and $C_1 = 2$ corresponds to the no return to isotropy which implies that b_{ij} remains unchanged.

To determine the functional dependence of C_1 we make use of the experimental data. Lumley and Newman (1977) have shown by using Comte-Bellot and Corrsin's data (1966) that in a vanishing small anisotropy, C_1 takes the form:

$$C_1 = 2 + 8\, R_\ell^{-1/2} \tag{2.40}$$

However, the linear behavior of C_1 in the limit $R_\ell \to \infty$ and $II \to o$ remains as an assumption. (See Figure 2.2). Chung (1978) attempted to reproduce the same result as in equation (2.40) and found that (in the limit $R_\ell \to \infty$ and $II \to 0$) the dependence of C_1 on II is stronger than that on R_ℓ. Nevertheless he accepted the basic assumption in (2.35) and expressed C_1 as follows:

$$C_1 = 2 + \gamma(II, III)/\mu(R_\ell) \tag{2.41}$$

where $\mu(R_\ell)$ is obtained by considering the average time scale of the return-to-isotropy over the spectral space - the approximate form being given by

$$\mu(R_\ell) = 3.16\ R^{-1/2} + 1.0 \qquad (2.42)$$

and where γ is a correction to the time scale due to the anisotropy. The functional dependence of π on II and III was determined by the use of the data of Comte-Bellot and Corrsin (1966), Mills and Corrsin (1959), and Uberoi (1956);the approximate form of γ is

$$\gamma = 110\ II\ \exp\ (-83\ II^{3/2})[1-2.47\ III^{1/3} + 2.24\ III] \qquad (2.43)$$

A careful examination of 12 different experiments of the Comte-Bellot and Corrsin data confirms Lumley and Newman findings if C_1 behaves linearly in the limit $R_\ell \to \infty$, see Figure 2.2. The fact that $C_1 \to 2$ when $R_\ell \to \infty$ and the data of Figure (2.2) support the validity of equation (2.38); the form for C_1 proposed by Lumley and Newman (1977) is given by:

$$C_1 = 2 + (\frac{1}{9} + 3\ III + II)\ F\,(R_\ell,\ II,\ III) \qquad (2.44)$$

The realizability condition must be imposed. $C_1 \to 2.0$ when either component of energy vanishes,or Schwarz's inequality is violated. It can be shown (Lumley 1978) that in order to satisfy the realizability condition we must have:

$$\frac{1}{9} + 3\ III + II \geq 0 \qquad (2.45)$$

The function $F(R_\ell,\ II,\ III)$ must be determined such that $F \to 0$ as $R_\ell \to 0$, $F \to 8.1\ R_\ell^{-.5}$ when $II \to 0$ and $R_\ell \to \infty$, and it fits the experimental data. The functional form that satisfies the realizability conditions and fits the data is given by:

$$C_1 = 2 + (\frac{1}{9} + 3\ III + II)\ \exp\ (-7.77/R_\ell^{1/2})[72/R_\ell^{1/2} + 80\ \ln\ (1 + 62.4\{-II + 2.3\ III\})] \qquad (2.46)$$

Equation (2.4) worked quite well in predicting the plane and axisymmetric wakes (see Taulbee and Lumley 1980), and it did as well in the present study. When equation (2.41) was used in predicting the axisymmetric jet ($R_\ell > 400$), however, there were no significant changes in the final result. Figure 2.4 shows that equation (2.41) is violating the realizability condition (2.45) since for a negative III, the first invariant II has a limiting value of $|II| < 1/12$ (Lumley 1978)

2.11 The Rapid Terms

The pressure-velocity gradient term of the pressure strain rate contribution can be obtained by solving equation (2.8) using Fourier transform for $p^{(1)}$.

Assuming a homogeneous mean field the rapid term can be written as

$$\overline{\frac{p^{(1)}}{\rho}(u_{i,j} + u_{j,i})} = 2U_{p,q}\,(I_{piqj} + I_{pjqi})\overline{q^2} \qquad (2.47)$$

where

$$I_{piq} = \int \frac{k_i k_p}{k^2} S_{qj}\, d\underline{k} \qquad (2.48)$$

and S_{qj} is the spectrum of the Reynolds stress.

An expression similar to equation (2.47) was first derived by Rotta (1951). A model for I_{piqj} has been proposed by several authors such as Hanjalic and Launder (1972), Launder, Reece and Rodi (1973) and Lumley (1975).

It is apparent from equation (2.47) that the rapid term arises from the interaction of the turbulence with the mean velocity gradients. The fourth order tensor I_{piqj} must satisfy the following constraints (Lumley 1978):

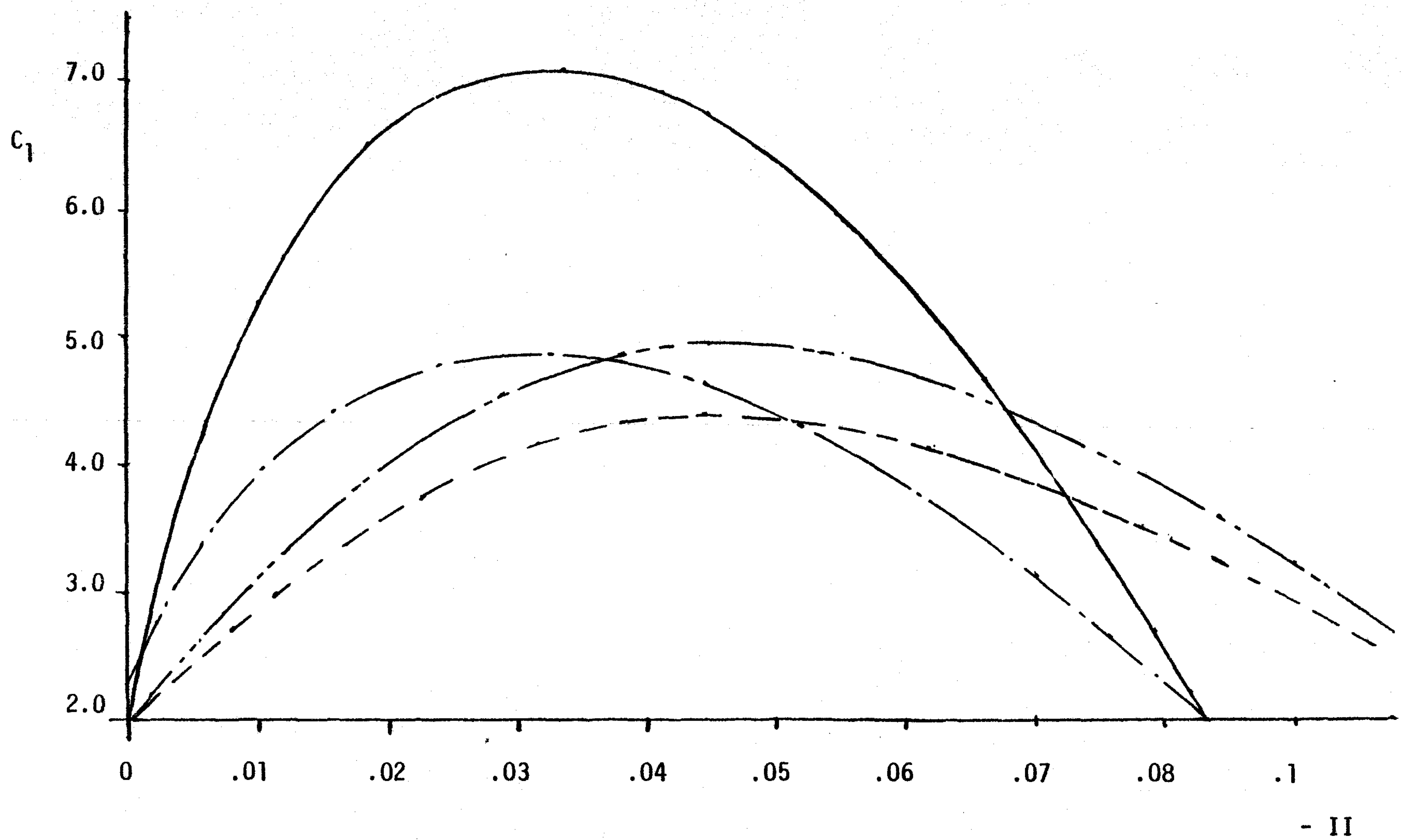

Figure 2.3   The Return to Isotropy Function C_1 for a Negative III.

(—, —·—) Lumley and Newman, (— -- —, ----) Chung (for R_ℓ = 4900, 100) respectively

Symmetry:

$$I_{piqj} = I_{pijq} \text{ and } I_{piqi} = I_{ipqi} \tag{2.48a}$$

Incompressibility:

$$I_{piij} = 0 \tag{2.48b}$$

Normalization:

$$I_{ppqi} = \overline{u_p u_j}/\overline{q^2} \tag{2.48c}$$

When the turbulence is isotropic, equation (2.48) can be integrated directly; hence (Crow, 1968, Rotta 1951a),

$$I_{pqij} = (4\delta_{ij}\delta_{pq} - \delta_{pi}\delta_{qi} - \delta_{pj}\delta_{qi})\ \overline{q^2}/30 \tag{2.49}$$

In general, I_{pqij} would depend at least on the anisotropic tensor b_{ij} and the Reynolds number. Hanjalic and Launder (1972) used a model for I_{pqij} with linear and quadratic terms in $\overline{u_i u_j}$. Later, Launder, Reece and Rodi (1975) dropped the quadratic terms, but Lumley and Khajeh-Nouri (1974b) also used a non-linear term. However, Lumley (1975a) argued that the model must be linear in $\overline{u_i u_j}$. Following Lumley (1978) we take the tensor I_{piqj} to be related to combinations which are linear in the anisotropic tensor b_{ij}.

A form which satisfies all the above requirements is given by

$$I_{piqj} = (4\delta_{pi}\delta_{qi} - \delta_{pq}\delta_{ij} - \delta_{pj}\delta_{qi})/30$$

$$- (b_{pi}\delta_{qi} - \delta_{pibqi})/3 + c[b_{pq}\delta_{ji} + b_{iq}\delta_{jp}$$

$$+ b_{pj}\delta_{qi} + b_{ij}\delta_{pq} - 11\ b_{pi}\delta_{qj}/3 - 4\delta_{pi}b_{qi}/3] \tag{2.50}$$

where c is an empirical constant.

The constant c should be evaluated using experimental data for a homogeneous turbulence from such experiments as of Tucker and Reynolds (1968) and Champagne, Harris and Corrsin (1970). Reynolds (1976) found that $c = -.1$ for the experiments of Tucker and Reynolds while $c = -.2$ worked better for the Champagne, Harris and Corrsin experiments; finally, he recommended $c = -.15$. Launder, Reece and Rodi (1975) use $c = -.145$. Taulbee and Lumley (1980) use $c = -.15$, and this value will be used in the present calculation as well.

2.12 Transport Terms

The remaining unknown correlations in the stress equation (2.12) are the diffusive transport terms. The contribution of the viscous terms to the diffusion is negligibly small for turbulent free shear flows, hence we will ignore that term. The transport by turbulent velocity fluctuations can be approximated by formulating a dynamic equation for $\overline{u_i u_j u_k}$.

Hanjalic and Launder (1972) simplified the exact equation for $\overline{u_i u_j u_k}$ to obtain the following:

$$\overline{u_i u_j u_k} = -\beta \frac{\overline{q^2}}{\varepsilon} [\overline{u_i u_\ell} (u_j u_k)_{,\ell} + \overline{u_j u_\ell} (u_k u_i)_{,\ell} + \overline{u_k u_\ell} (u_i u_j)_{,\ell} \tag{2.51}$$

where β is a fixed constant. However, several computations (Rodi 1972, Launder and Morse 1978) have been carried out using the simpler model proposed by Daly and Harlow (1970); namely,

$$\overline{u_i u_j u_k} = -\beta \frac{\overline{q^2}}{\varepsilon} \overline{u_k u_\ell} (u_i u_j)_{,\ell} \tag{2.52}$$

Lumley (1978a) argued that homogeneous turbulence is observed to be Gaussian in the energy containing range, even in the presence of non-zero velocity gradients, and that departure from Gaussian behavior is associated with inhomogeneity. This is consistent with the fact that the fluxes $\overline{u_i u_j u_k}$ cannot be non-zero if the turbulence is Gaussian.

By assuming a weakly inhomogeneous turbulence and performing an order of magnitude analysis of the exact equations for $\overline{u_i u_j u_k}$, it can be shown that (see Lumley 1978a):

$$\overline{u_i u_j u_k} = -\frac{2}{3C_1(2-b)} \frac{\overline{q^2}}{\varepsilon} [G_{ijk} + \frac{C_1 - 2}{(4-9b/2)C_1+10} (\delta_{ij}G_k + \delta_{ik}G_j + \delta_{jk}G_i)] \tag{2.53}$$

and

$$\overline{q^2 u_i} = \frac{3}{(4-9b/2)C_1+10} \frac{\overline{q^2}}{\varepsilon} G_i \tag{2.54}$$

where

$$G_{ijk} = \overline{u_k u_p}\,(\overline{u_i u_j})_{,p} + \overline{u_j u_p}\,(u_i u_k)_{,p} + \overline{u_i u_p}\,(\overline{u_j u_k})_{,p} \tag{2.55}$$

and

$$G_k = \overline{u_k u_p}\,\overline{q^2}_{,p} + 2\,\overline{u_q u_p}\,(\overline{u_q u_k})_{,p} \tag{2.56}$$

C_1 is the same coefficient given by equation (2.39) and b is an arbitrary constant. The only condition on b is that b<1 in order for this model to relax to Gaussian. Taulbee and Lumley (1980) took b=0 but indicated that a b slightly greater than zero would give a better agreement with the experiments. Notice that if $i \neq j \neq k$ equation (2.53) reduces to the model (2.51) if

$$\beta = \frac{2}{3C_1(2-b)} \tag{2.57}$$

The pressure diffusive transport has been neglected in many recent computations. Bradshaw and Ferries (1965) indicated that the measured energy balance closes quite well when the pressure-diffusion term in their k - equation is neglected. Rodi (1972), Hanjalic and Launder (1975) and Launder and Morse (1972) have totally neglected the pressure transport in their models because of lack of evidence of its importance. In order to account for the pressure diffusion we assume homogeneous flow and Fourier transform equation (2.9). The second (return to isotropy) part of the pressure is then given by;

$$-\,[p^{(2)}]/\rho = \left(\frac{\kappa_q \kappa_j}{\kappa^2}\right)[u_i u_j] \qquad (2.58)$$

where [ ] denotes Fourier transform.

Multiplying equation (2.58) by u_k,averaging, and integrating the result yields

$$-\,\overline{p^{(2)}}\,u_k = \int \left(\frac{\kappa_i \kappa_j}{\kappa^2}\right) S_{ij\kappa}\, d\underline{\kappa} \qquad (2.59)$$

where $S_{ij\kappa}$ is the spectrum of $\overline{u_i u_j u_k}$.

We will define

$$I_{ijpqr} = \int \frac{\kappa_i \kappa_j}{\kappa^2} S_{pqr}\, d\underline{\kappa} \qquad (2.60)$$

and attempt to express the integral as in linear combination of the triple velocity correlation. It follows from symmetry that:

$$I_{ijpqr} = I_{jipqr} \quad \text{and} \quad I_{ijpqr} = I_{ijqpr}$$

For incompressible flow we further require:

$$I_{ijpqj} = 0 \quad \text{and} \quad I_{iippr} = \overline{u_p u_p u_r}$$

The most general linear form of $u_i u_j u_k$ contains five coefficients. However by applying the above conditions all the coefficients can be determined and we obtain:

$$I_{ijpqr} = \frac{2}{5}\,\delta_{ij}\,\overline{u_p u_q u_r} - \frac{1}{10}(\delta_{ir}\,\overline{u_j u_p u_q} + \delta_{jr}\,\overline{u_i u_p u_q}\,) \qquad (2.61)$$

Hence,

$$-\,\frac{\overline{p^{(2)}}}{\rho}\,u_r = \frac{1}{5}\,\overline{q^2 u_r} \qquad (2.62)$$

where $\overline{q^2 u_r}$ is given by equation (2.54).

2.13 Transport Terms for ε

It is not possible to write an exact equation for the transport of dissipation, and equation (2.14) has been proposed by an analogy with the transport equations of velocity and temperature variance. Hence, we have to model the transport flux $\overline{\varepsilon u_k}$ by analogy with the transport term for Reynolds stress.

By assuming that $\overline{q^2}^2/\varepsilon$ does not vary too much across the width of the flow (i.e., across the jet) and that if ε vanishes, $\overline{q^2}$ also must vanish such that $\overline{q^2}^m/\varepsilon$ remains bounded so that

$$\overline{q^2}^m \sim \varepsilon \quad , \tag{2.63}$$

it follows that

$$m\varepsilon \overline{q^2}_{,k} = \overline{q^2}\, \varepsilon_{,k} \tag{2.64}$$

Thus the analogue to equation (2.64) is:

$$\overline{q^2}\, (\overline{\varepsilon u_k})_{,k} = m\varepsilon[(\overline{q^2} + 2P/\rho)u_k]_{,k} \tag{2.65}$$

where we have included the pressure transport in the right side of equation (2.65).

From equation (2.65) we can again see by analogy that if the transport of $\overline{q^2}$ can be modeled as in section (2.10), the transport of ε can be modeled in a similar way. Using equation (2.62) for the pressure we obtain

$$\varepsilon u_k = \frac{3}{5} m(\varepsilon/\overline{q^2})\, \overline{q^2 u_k} \tag{2.66}$$

Substituting equation (2.54) into equation (2.66) we obtain for the dissipation flux:

$$\overline{\varepsilon u_k} = - \frac{.9}{(2-9b/4)C_1+5} \left(\frac{\overline{q^2}}{\varepsilon}\right) [u_k u_p + 2\overline{u_i u_k}\; \overline{u_i u_p}/\overline{q^2}]\varepsilon_{,p} \tag{2.67}$$

where we have replaced $\overline{u_i u_{j,k}}$ in equation (2.51) by $(\overline{u_i u_j}/m\varepsilon)\overline{\varepsilon}_{,k}$

We have to note that equation (2.67) does not introduce a new constant since C_1 is the same parameter obtained in the return to isotropy part.

This completes the Reynolds stress closure which consists now of the equations for the mean motion; equations (2.1) and (2.2), and the six equations for the Reynolds stresses components, three normal and three tangential components, and the equation for the dissipation rate of energy.

2.14 The Final Form of the Reynolds Stress Closure.

The Reynolds stress equation is now given by

$$\dot{\overline{u_i u_j}} + U_k(\overline{u_i u_j})_{,k} = -(\overline{u_k u_i}\, U_{j,k} + \overline{u_k u_j}\, U_{i,k})$$

$$[\frac{2}{3(2-b)C_1}\frac{\overline{q^2}}{\varepsilon} G_{ijk} - \frac{1}{(2-9b/4)C_1+5}\frac{\overline{q^2}}{\varepsilon}\{\frac{C_1-2}{3(2-b)C_1} G_k\delta_{ij}$$

$$+ (\frac{C_1-2}{3(2-b)C_1} - \frac{3}{10})(G_j\delta_{ik} + G_i\delta_{jk})\}]_{,k} - C_1\bar{\varepsilon}\, b_{ij}$$

$$+ 2U_{p,q}\,(I_{piqj} + I_{pjqi})\overline{q^2} - \frac{2}{3}\varepsilon\delta_{ij} \tag{2.68}$$

where

$$G_{ijk} = \overline{u_k u_p}\,(\overline{u_i u_j})_{,p} + \overline{u_j u_p}\,(\overline{u_i u_k})_{,p} + \overline{u_i u_p}\,(\overline{u_j u_k})_{,p} \tag{2.69}$$

$$G_k = \overline{u_k u_p}\,\overline{q^2}_{,p} + 2\overline{u_i u_p}\,(u_i u_k)_{,p} \tag{2.70}$$

and

$$I_{piqj} = -\frac{1}{3}(b_{pi}\delta_{qj} - \delta_{pi}b_{qj}) + \frac{1}{30}(4\delta_{pi}\delta_{qj} - \delta_{pq}\delta_{ij} - \delta_{pj}\delta_{qi})$$

$$+ c[b_{pq}\delta_{ji} + b_{iq}\delta_{jp} + b_{pj}\delta_{qi} + b_{ij}\delta_{pq} - \frac{11}{3} b_{pi}\delta_{qj} - \frac{4}{3}\delta_{pi}b_{qi}] \tag{2.71}$$

$$b_{ij} = \frac{\overline{u_i u_j}}{\overline{q^2}} - \frac{1}{3}\delta_{ij} \tag{2.72}$$

The dissipation equation takes the form

$$\dot{\overline{\varepsilon}} + U_k\overline{\varepsilon}_{,k} = \left[\frac{.9}{(2-9b/4)C_1+5}\,\frac{\overline{q^2}}{\varepsilon}\,\{\overline{u_k u_k} + 2\,\overline{u_i u_k}\,\frac{\overline{u_i u_k}}{\overline{q^2}}\}\,\overline{\varepsilon}_{,k}\right]_{,k} \qquad (2.73)$$

$$- \psi_1\,\overline{\varepsilon}\,b_{ij}\,U_{i,j} - \psi_o\,\frac{\overline{\varepsilon^2}}{q^2}$$

The above system of equations consists of 4 equations for the mean flow and 6 equations for the Reynolds stress components and an equation for the dissipation rate of energy. This set of 11 equations is sufficient to solve for the eleven unknowns, namely
U, V, W, p, $\overline{u^2}$, $\overline{v^2}$, $\overline{w^2}$, $\overline{uv}$, $\overline{uw}$, $\overline{vw}$ and ε.

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CHAPTER 3

The Two Equation Models

3.1 Introduction

The two equation models employ two partial differential equations of turbulence in addition to the governing equations of the mean motion. For example, the model of Jones and Launder (k-ε model) solves for k, the kinetic energy of turbulence and ε, the dissipation rate of the kinetic energy of turbulence, while that of Rodi and Spalding solves for k and kℓ in their (k-kℓ) model where k is the same as above and ℓ is a characteristic length scale of turbulence. Spalding (1972) proposed the k-ω model which is similar to the above where he replaces ε by $(\frac{\varepsilon}{k})^2$. These models, among others of their class, showed some success in the early 70's in predicting the turbulent flow field, in high Reynolds number flows such as mixing layers, turbulent jets and near wakes. Less success has been reported where these models were applied near wall regions or in far-field jets or wakes (Launder 1975).

In the early stage of this study a great amount of time has been spent on the applications of the standard version of the k-ε model. Most of the calculations have been carried out for the turbulent free jets (plane and axisymmetric jets). The primary reason for using the k-ε model was to test several numerical schemes which were developed for use in solving the system of equations in the Reynolds stress closure (chapter 4.)

3.2 The Eddy Viscosity Concept

Before we turn our attention to the k-ε model we must familiarize ourselves with eddy viscosity concept since it is an essential ingredient in the majority of the two equation models, which do not solve for the shear stress $\overline{uv}$ directly. This concept assumes a simple proportionality relation between turbulent transport and mean velocity gradient. The proportionality constant is the eddy viscosity ν_t. For turbulent free shear flows ν_t is defined as (Boussinesq 1877)

$$\overline{uv} = \nu_t \frac{\partial U}{\partial y} \tag{3.1}$$

Here ν_t is not a property of the fluid as in the laminar case, but depends solely on the state of turbulence. Hence ν_t can vary from one flow to another, and also it may vary across the flow.

The zero equation model which employs this principle (see Appendix B) has had a variety of success and failures in predicting turbulent flow fields. Tennekes and Lumley (1972) argue that the eddy viscosity models are expected to be successful when the turbulent flow is characterized by single time and length scales. This suggests that in the similarity regions (the far field) of turbulent free shear flows, this simple model should provide a reasonable prediction of the mean velocity profile and in turn the shear stress $\overline{uv}$. In the k-ε model formulation it is natural to assume that the eddy viscosity will depend on the intensity of turbulence through k, the kinetic energy of turbulence, and ε, the dissipation rate. Therefore we assume:

$$\nu_t \sim k^2/\varepsilon \tag{3.2}$$

or

$$\nu_t = C_\mu \frac{k^2}{\varepsilon} \tag{3.3}$$

where C_μ is a proportionality constant that will be determined.

3.3 The k-ε Closure Model

The k-ε closure consists of the equations for the mean flow,(2.1) and (2.2), the kinetic energy equation,(2.4),and the equation for the dissipation rate of energy,(2.6). For high Reynolds number flow with constant density and viscosity the system of equations can be written as follows:

$$U_{i,i} = o \tag{3.4}$$

$$\dot{U}_i + U_j U_{i,j} = -\frac{1}{\rho} P_{,i} + [\nu U_{i,j} - \overline{u_i u_j}]_{,j} \tag{3.5}$$

$$\dot{k} + U_j k_{,j} = -[u_j(k + p/\rho)]_{,j} - \overline{u_i u_j}\, U_{i,j} - \varepsilon \tag{3.6}$$

$$\dot{\varepsilon} + U_j \varepsilon_{,j} = -[\overline{u_j \varepsilon}]_{,j} - C_{\varepsilon 1} \frac{\varepsilon}{k} \overline{u_i u_j}\, U_{i,j} - C_{\varepsilon 2} \frac{\varepsilon^2}{k} \tag{3.7}$$

where

$$k = \frac{\overline{u_i u_j}}{2} \tag{3.8}$$

$$\varepsilon = 2\nu\, \overline{u_{i,j} u_{i,j}} \tag{3.9}$$

and $C_{\varepsilon 1}$ and $C_{\varepsilon 2}$ are the model constants.

The diffusion terms, the first terms on the right hand side of equations (3.6) and (3.7), will be modeled in a much simpler way than was done in Chapter 2. It will be simply assumed that k and ε diffuse down their gradients. Thus, we assume

$$u_j(k + P/\rho) = \frac{\nu_t}{\sigma_k} k_{,j} \tag{3.10}$$

and

$$\overline{u_j \varepsilon} = \frac{\nu_t}{\sigma \varepsilon} \varepsilon_{,j} \tag{3.11}$$

where σ_k and σ_ε are constants to be determined from experiments,and ν_t is the eddy viscosity given by equation (2.3) which is determined by the calculations. Hence with equations (3.10) and (3.11) the system equations (3.4)-(3.7) constitutes a closed set which is applicable to any turbulent free shear flow.

The continuity equation (3.9) and momentum equation (3.5) are written in Cartesian form in Appendix A. In the momentum equation only the first order terms will be retained. The estimate of the neglected terms amount to only about 8% of the total momentum transport when integrated across the flow. This particular point will be discussed in Chapter 4 when we solve for all the components of the Reynolds stress.

3.4 The Final Form of the k-ε Model

Now let us consider the application of the k-ε model to boundary free shear flows by introducing the following assumptions:

i. Steady motion ($\frac{\partial}{\partial t} = o$)

ii. High Reynolds number flow ($\mu \frac{\partial^2 U}{\partial y^2} << \frac{\partial \overline{uv}}{\partial y}$)

iii. Derivatives with respect to x are negligible compared to those with respect to y.

iv. The flow field is far away from the source ($k^2/\varepsilon \approx$ constant).

v. The W component of the mean velocity is zero and $\frac{\partial}{\partial z} = o$, (in the axisymmetric case $\frac{\partial}{\partial \theta} = o$).

Based on these assumptions the final form of the k-ε closure model is given by:

Continuity Equation:

$$\frac{\partial U}{\partial x} + \frac{1}{y^i} \frac{\partial y^i V}{\partial y} = o \qquad (3.12)$$

Streamwise Momentum (see Appendix A)

$$U \frac{\partial U}{\partial x} + V\frac{\partial U}{\partial y} = - \frac{1}{y^i} \frac{\partial}{\partial y} (y^i \overline{uv}) \tag{3.13}$$

Turbulence Viscosity Hypothesis

$$-\overline{uv} = \nu_t \frac{\partial U}{\partial y} \tag{3.14}$$

$$\nu_t = C_\mu \frac{k^2}{\varepsilon} \tag{3.15}$$

Turbulent Kinetic Energy

$$U \frac{\partial k}{\partial x} + V \frac{\partial k}{\partial y} = \frac{1}{y^i} \frac{\partial}{\partial y} (y^i \frac{\nu_t}{\sigma_k} \frac{\partial k}{\partial y}) + \nu_t (\frac{\partial U}{\partial y})^2 - \varepsilon \tag{3.16}$$

Turbulent "Dissipation" Rate

$$U \frac{\partial \varepsilon}{\partial x} + V\frac{\partial \varepsilon}{\partial y} = \frac{1}{y^i} \frac{\partial}{\partial y} (y^i \frac{\nu_t}{\sigma_\varepsilon} \frac{\partial \varepsilon}{\partial y}) + C_{\varepsilon 1} \frac{\varepsilon}{k} \nu_t (\frac{\partial U}{\partial y})^2 - C_{\varepsilon 2} \frac{\varepsilon^2}{k} \tag{3.17}$$

where i=o for the plane jet and i=1.0 for the axisymmetric turbulent jet.

The model constants as originally proposed by Jones and Launder (1972) for high Reynolds number flow are given in Table 3.1.

C_μ	$C_{\varepsilon 1}$	$C_{\varepsilon 2}$	σ_k	σ_ε
.09	1.55	2.0	1.0	1.3

Table 3.1 Empirical values of the k-ε model constants as suggested by Jones and Launder (1972).

3.5 Similarity Formulation

An important feature of free turbulent shear flows is their tendency to become self-similar after certain development regions. As mentioned earlier, this is consistent with the fact that their motions are characterized by single velocity and length scales.

Consider either a symmetrical, two-dimensional or an axisymmetrical, three-dimensional jet (Figure 3.1). For the velocity scale we chose the centerline value of the mean velocity U_m. For the length ℓ at which the mean velocity U is half its maximum. Hence for self-preserving flow we define:

$$U = U_m f(\eta) \tag{3.18}$$

and

$$V = U_m h(\eta) \tag{3.19}$$

where

$$\eta = \frac{y}{\ell(x)} \tag{3.19a}$$

Further, the components of the kinematic turbulence stress tensor can be expressed as:

$$\begin{aligned} \overline{u^2} &= U_m^2\, g_1(\eta) \\ \overline{v^2} &= U_m^2\, g_2(\eta) \\ \overline{w^2} &= U_m^2\, g_3(\eta) \\ \overline{uv} &= U_m^2\, g_4(\eta) \end{aligned} \tag{3.20}$$

Similarly, the kinetic energy of the turbulence k and the dissipation rate ε, assume the following form:

$$\begin{aligned} k &= U_m^2\, K(\eta) \\ \varepsilon &= \frac{U_m^3}{\ell(x)} E(\eta) \end{aligned} \tag{3.21}$$

Substituting (3.18) and (3.19) into (3.12), the continuity equation becomes:

$$\frac{\ell}{U_m} \frac{dU_m}{dx} f(\eta) - \frac{d\ell}{dx} \eta\, f'(\eta) + \frac{1}{\eta^i} (\eta^i h)' = o \tag{3.22}$$

Similarly, if we substitute (3.18)-(3.20) into equation (3.13) the momentum equation becomes:

$$\frac{\ell}{U_m}\frac{dU_m}{dx} ff' - \frac{d\ell}{dx}\eta ff' + hf' + \frac{1}{\eta^i}(\eta^i g_4)' + \frac{\ell}{2U_m}\frac{dU_m}{dx}(g_1-g_2)$$

$$- \frac{d\ell}{dx}(g_1'-g_2')\eta = o \qquad (3.23)$$

where the prime denotes differentiation with respect to η.

Now for the flow to be self-preserving (self-similar) the equations (3.22) and (3.23) must be independent of x. Hence the coefficients involved in these equations must be constants. Therefore for similarity solutions we must have:

$$\frac{\ell}{U_m}\frac{dU_m}{dx} = a_o \qquad (3.24)$$

$$\frac{d\ell}{dx} = a_1 \qquad (3.25)$$

where a_o and a_1 are constants.

Equation (3.25) suggests that in the self-preservation region the turbulent jets (plane or axisymmetric) spread linearly with x. That is,

$$\ell = a_1 x \qquad (3.26)$$

Equation (3.24) is satisfied if U_m behaves like $U_m \sim x^n$ for any value of n. To determine the proper value for the exponent n we impose the momentum integral constraint, which has to be satisfied at any cross section x. From Appendix A, the momentum integral constraint reads

$$2\,\pi^i \int_o^\infty [U^2 + \overline{u^2} - \frac{\overline{v^2}+\overline{w^2}}{2}]\, y^i dy = M_o/\rho \qquad (3.27)$$

where

$$M_o = \pi^i U_o^2 \left(\frac{d_o}{i+1}\right)^{i+1} \qquad (3.28)$$

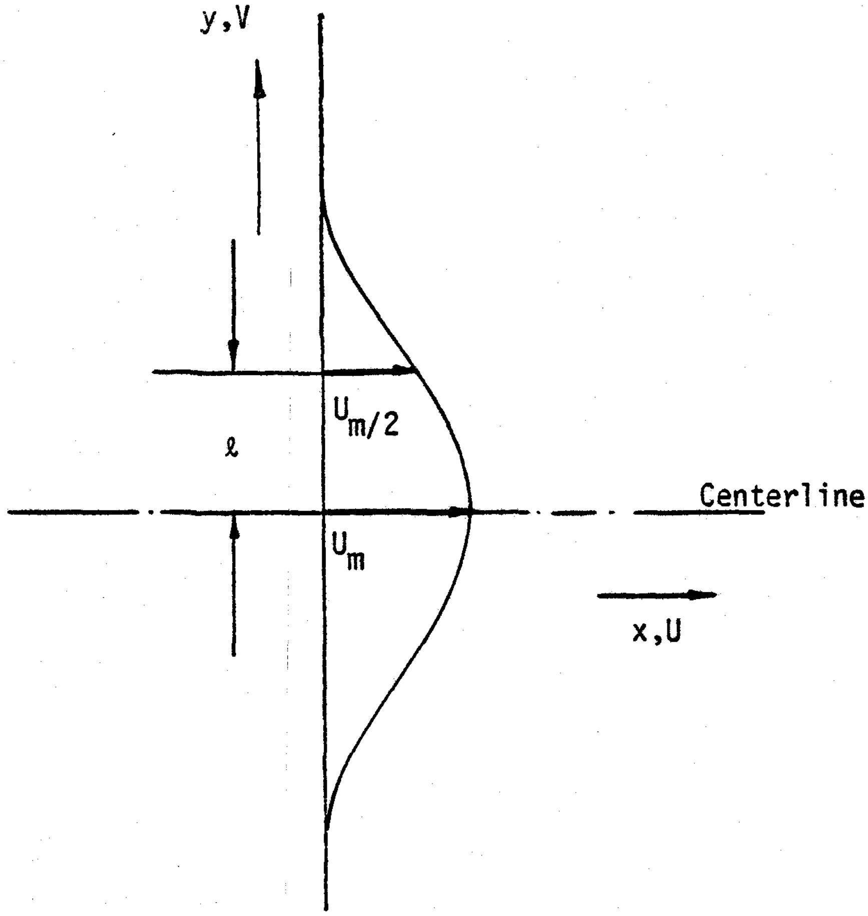

Figure 3.1 Symmetrical Turbulent Jet in Still Surrounding.

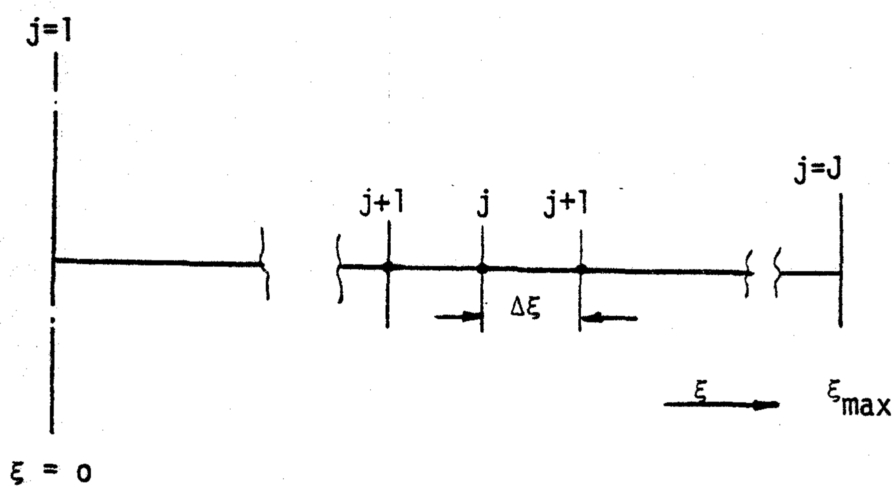

Figure 3.2 Finite Grid.

In similarity variables the momentum integral becomes:

$$\int_0^\infty [f^2 + g_1 - \frac{g_2+g_3}{2}]\, \eta^i\, d\eta = \frac{U_0^2}{2U_m^2} \frac{1}{\ell^{i+1}} (\frac{d_0}{i+1})^{i+1} \tag{3.29}$$

In order for the right hand side of equation (3.29) to be a constant we must have;

$$U_m^2\, \ell^{i+1} = \text{constant} \tag{3.30}$$

Hence from equation (3.26) and (3.30) the centerline velocity of self-preserving turbulent jet must have the following decay law:

$$U_m \sim x^{-\frac{i+1}{2}} \tag{3.31}$$

The constant of proportionality in equations (3.26) and (3.31) will be determined based on experimental data of the plane and axisymmetric jet respectively.

Now the continuity and momentum equations take the form:

$$- a_1[\frac{i+1}{2} f(\eta) + \eta f'(\eta)] + \frac{1}{\eta^i} (\eta^i h(\eta))' = o \tag{3.32}$$

$$- a_1[\frac{i+1}{2} f(\eta)^2 + \eta f(\eta)\, f'(\eta)] + h(\eta)\, f'(\eta) - \frac{1}{\eta^i}[\eta^i g_4(\eta)]'$$

$$- a_1(i+1)[g_1(\eta) - g_2(\eta)] - a_1[g_1'(\eta) - g_2'(\eta)]\eta = o \tag{3.33}$$

where a_1 is a constant for a particular flow.

The continuity equation can be integrated directly to give

$$h(\eta) = \frac{a_1}{\eta^i} [\eta^{i+1} f - \frac{i+1}{2} \int_0^\eta \eta^i f d\eta] \tag{3.34}$$

If we substitute equation (3.34) into equation (3.33), the momentum equation becomes:

$$a_1 \frac{i+1}{2} [f^2 + \frac{1}{\eta^i} f' \int_0^\eta \eta^i f d\eta] + \frac{1}{\eta^i} [C_\mu \frac{K^2}{E} \eta^i f']'$$

$$+ a_1 \frac{(i+1)}{4} [g_1 - g_2] + a_1 [g_1' - g_2'] \eta = 0 \qquad (3.35)$$

where we have used the eddy viscosity hypothesis (3.14) and (3.15); that is,

$$g_4 = - C_\mu \frac{K^2}{E} f' \qquad (3.36)$$

If we substitute (3.18) and (3.21) into equation (3.16) and (3.17) and use equation (3.24) to eliminate the cross stream component of the mean velocity, the energy and dissipation rate equations become:

$$(\frac{C_\mu}{\sigma_k} \frac{K^2}{E} \eta^i K')' + a_1 \frac{i+1}{2} [K' \int_0^\eta \eta^i f d\eta + 2\eta^i K f] + C_\mu \frac{K^2}{E} \eta^i f'^2 - \eta^i E = 0 \qquad (3.37)$$

$$(\frac{C_\mu}{\sigma_\varepsilon} \frac{K^2}{E} \eta^i E')' + a_1 \frac{i+1}{2} [E' \int_0^\eta \eta^i f d\eta + (5-i)\ \eta i E f)$$

$$+ C_{\varepsilon 1} (\frac{E}{K})(C_\mu \frac{K^2}{E})\ \eta^i\ f'^2 - C_{\varepsilon 2} \frac{E^2}{K} \eta^i = 0 \qquad (3.38)$$

Now let us define the term involving the integral in the above equations as follows:

$$G(\eta) = \int_0^\eta \eta^i f(\eta) d\eta \qquad (3.39)$$

Further the momentum integral equation (3.29) can be written as

$$P(\eta) = \int_0^\eta f^2(\eta)\ \eta^i\ d\eta \qquad (3.40)$$

where we have neglected the turbulence contribution to the momentum integral and we have the following conditions on $P(\eta)$:

$$P(o) = o \tag{3.41a}$$

$$P(\infty) = \frac{U_o^2}{2U_m^2} \frac{1}{\ell^{i+1}} \left(\frac{d_o}{i+1}\right)^{i+1} \tag{3.41b}$$

The parameter a_1 which represents the spreading rate of the jets is a scaling factor in the above system of equations and it can be eliminated if we define a new variable such that:

$$\xi = \sqrt{a_1}\eta \tag{3.42a}$$

$$d = \sqrt{a_1}d_o \tag{3.42b}$$

where d is the jet diameter. If we write equation (3.31) as

$$U_m = C\ U_o\ \left(\frac{d}{x}\right)^{\frac{i+1}{2}} \tag{3.42c}$$

where C is am empirical constant,then the second condition (3.41b) becomes

$$P(\infty) = \frac{1}{2(i+1)^{i+1}\ C^2} \tag{3.43}$$

The final form of the k-ε closure model for the self-preserving jet is now given by the following system ordinary differential equations:

$$\underset{(a)}{[(C_\mu \frac{K^2}{E})\ \xi^i f']'} + \underset{(b)}{\frac{i+1}{2}\ [Gf' + \xi^i f^2]} = o \tag{3.44a}$$

$$\underset{(a)}{[(C_\mu \frac{K^2}{E})\ \frac{\xi^i}{\sigma_k}\ K']'} + \underset{(b)}{\frac{i+1}{2}\ [GK' + 2\xi^i Kf]} + \underset{(c)}{(C_\mu \frac{K^2}{E})\xi^i f'^2} - \underset{(d)}{\xi^i E} = o \tag{3.44b}$$

$$\underset{(a)}{[(C_\mu \frac{K^2}{E})\ \frac{\xi^i}{\sigma_\varepsilon}\ E']} + \underset{(b)}{\frac{i+1}{2}\ [GE' + (5-i)\ \xi i\ Ef]}$$

$$+ \underset{(c)}{C_{\varepsilon 1}(C_\mu \frac{K^2}{E})\ \frac{E}{K}\ \xi^i f'^2} - \underset{(d)}{\xi i\ C_{\varepsilon 2} \frac{E^2}{K}} = 0 \tag{3.44c}$$

$$P' - \xi^{i} f^{2} = o \qquad (3.44d)$$

$$G' - \xi^{i} f = o \qquad (3.44e)$$

Here we have expressed the integrals which result from the continuity equation and from the momentum running integral in differential form - equations (3.44d) and (3.44e)- so that the above system of five equations can be solved simultaneously. Hence with these constraints both the momentum and the continuity equations will be automatically satisfied.

The terms in the above equations can be identified as:

a) Turbulence diffusive transport.

b) Convection or advection.

c) Production by the mean motion.

d) Mechanical dissipation or destruction.

Boundary Conditions

The only constraint which has been imposed on the system of equations is that the requirement that the non-dimensionalized mean velocity at line of symmetry ($\xi=0$) be equal to unity. Since the exact values of the energy and the dissipation at $\xi=0$ are not known, we can only assume that their derivatives at the line of symmetry are zero. At the outer edge ($\xi\to\infty$) we require that the functions f, K, and E and their derivatives vanish. The equations (3.44d)and (3.44e)are first order differential equations, and the obvious conditions on P and G are that they approach zero as $\xi\to 0$. At the outer edge the equations for P and G will be evaluated from the difference equations.

Based on these conditions the above equations will be written in finite difference form and evaluated at the centerline. Hence at the centerline of the jet (ξ=0) the system of equations (3.44) becomes:

$$2(C_\mu \frac{K^2}{E})\, f'' + f^2 = 0 \tag{3.45a}$$

$$(i+1)(\frac{C_\mu}{\sigma_k} \frac{K^2}{E})\, k'' + (i+1)\, Kf - E = 0 \tag{3.45b}$$

$$(i+1)(\frac{C_\mu}{\sigma_\varepsilon} \frac{K^2}{E})E'' + (i+1)\frac{(5-i)}{2}\, Ef - C_{\varepsilon 2} \frac{E^2}{K} = 0 \tag{3.45c}$$

$$P(0) = 0 \tag{3.45d}$$

$$G(0) = 0 \tag{3.45e}$$

Recall that i=0 corresponds to the plane jet while i=1 for the axisymmetric case.

3.6 Quasi-Linearization

An analytical solution of the system (3.44) has not been obtained, and the best that can be done for now is to obtain a numerical one. For most practical purposes of engineering interest, a numerical solution will provide the required information with a fair degree of accuracy. However, numerical instabilities which arise mainly from the non-linearity of the differential equations are of major concern since they can prevent any reasonable solution of the system. Also the above system of equations are strongly coupled and this may contribute to instability. To overcome these difficulties the equations will be linearized, and then iterations will be carried out on this linearized set.

In order to do this let us consider the case of a two-dimensional jet (i.e.,i=0); the treatment of the axisymmetric case will be quite similar to the two dimensional case. For the two-dimensional case we let i=0 in (3.44) to get:

$$(Df')' + \frac{1}{2}(Gf' + f^2) = 0 \tag{3.46a}$$

$$(\frac{D}{\sigma_K} K')' + \frac{1}{2}(GK' + 2Kf) + Df'^2 - E = 0 \tag{3.46b}$$

$$(\frac{D}{\sigma_\varepsilon} E')' + \frac{1}{2}(GE' + 5Ef) + C_{\varepsilon 1} D\frac{E}{K} f'^2 - C_{\varepsilon 2} \frac{E^2}{K} = 0 \tag{3.46c}$$

$$P' - f^2 = 0 \tag{3.46d}$$

$$G' - f = 0 \tag{3.46e}$$

where

$$D = C_\mu \frac{K^2}{E} \tag{3.47}$$

The non-linear terms will be expanded in a Taylor series about some known values and only the first order quantities will be retained. For example, let us consider the last terms in the momentum equation (3.46a). They will be linearized as follows:

$$f^2 = f_o^2 + 2f_o (f-f_o) \tag{3.48a}$$

$$Gf' = G_o f'_o + G_o(f'-f'_o) + f'_o (G-G_o) \tag{3.48b}$$

where the quantities with the "o" subscript are assumed to be a constant and in this case they are known from previous iterations. Hence the equations (3.46) can be written in the following linearized form:

$$(Df')' + \frac{G_o}{2} f' + f_o f + \frac{f'_o}{2} G = \frac{1}{2} (f^2 + Gf')_o \tag{3.49a}$$

$$(\frac{D}{\sigma_K} k') + \frac{G_o}{2} K' + f_o K + K_o f + K'_o G + (2Df')_o f' - E$$

$$= \frac{1}{2} (2Kf + GK')_o + (Df'^2)_o \tag{3.49b}$$

$$\left(\frac{D}{\sigma_\varepsilon} E'\right)' + \frac{G_o}{2} E' + \left(\frac{5f}{2} - C_\mu \frac{C_{\varepsilon 2}K}{D}\right)_o E + (2C_\mu C_{\varepsilon 1} K f')_o f' +$$

$$+ \frac{5E_o}{2} f + \left(C_\mu C_{\varepsilon 1} f'^2 - \frac{C_{\varepsilon 2} C_\mu E}{D}\right)_o K + \frac{E_o'}{2} G = \frac{1}{2}(5Ef + GE')_o$$

$$+ C_\mu \left(C_{\varepsilon 1} f'^2 K - \frac{C_{\varepsilon 2}}{D} EK\right)_o \quad (3.49c)$$

$$P' - 2f_o f = -f_o^2 \quad (3.49d)$$

$$G' - f = o \quad (3.49e)$$

3.7 The "finite" Difference Equations

Using a control differencing scheme (see Figure 3.2), which is of a second order accuracy, the differential equations (3.49) can be expressed in finite difference form. After some algebraic rearrangement of the various terms the following difference equations results:

$$\left[\frac{D_{j-1}+D_j}{2\Delta\xi^2} - \frac{G_j}{2\Delta\xi}\right]_o F_{j-1} + \left[f_j - \frac{4D_j}{2\Delta\xi^2}\right]_o f_j + \left[\frac{D_{j+1}+D_j}{2\Delta\xi^2} + \frac{G_j}{2\Delta\xi}\right] f_{j+1}$$

$$+ \left(\frac{f_j'}{2}\right)_o G_j = (f_j^2 + G_j f_j')_o \quad (3.50a)$$

$$\left[\frac{D_{j-1}+D_j}{2\sigma_K\Delta\xi^2} - \frac{G_j}{2}\frac{1}{2\Delta\xi}\right]_o K_{j-1} + \left[f_j - \frac{4D_j}{2G_K\Delta\xi^2}\right] + \left[\frac{D_{j+1}+D_j}{2G_K\Delta\xi^2} + \frac{G_j}{2}\frac{1}{2\Delta\xi}\right] K_{j+1}$$

$$- \left(\frac{D_i f_j'}{\Delta\xi}\right)_o f_{j-1} + (K_j)_o f_j + \left(\frac{D_j f_j'}{\Delta\xi}\right)_O f_{j+1}$$

$$+ \left(\frac{K_j'}{2}\right)_o G_j - E_j = \frac{1}{2}[2K_j f_j + G_j K_j]_o + (D_j f_j'^2)_o \quad (3.50b)$$

$$\left[\frac{D_{j-1}+D_j}{2\sigma_\varepsilon\Delta\xi^2} - \frac{G_j}{2}\frac{1}{2\Delta\xi}\right]_o E_{j-1} + \left[\frac{5}{2} f_j - \frac{4D_j}{2\sigma_\varepsilon\Delta\xi^2} - \frac{C_\mu C_{\varepsilon 1} K_j}{D_j}\right]_o E_j$$

$$+ \left[\frac{D_{j+1}+D_j}{2\sigma_\varepsilon\Delta\xi^2} + \frac{G_j}{2}\frac{1}{2\Delta\xi}\right]_o E_{j+1} - \left(\frac{C_\mu C_{\varepsilon 1} f_j' k_i}{\Delta\xi}\right)_o f_{j-1}$$

$$+ \frac{5}{2} (E_j)_o f_j + (\frac{C_\mu C_{\varepsilon 1} f'_j K_j}{\Delta\xi})_o f_{j+1}$$

$$+ [C_\mu C_{\varepsilon 1} f_j'^2 - \frac{C_\mu C_{\varepsilon 2} E_j}{D_j}]_o K_j + \frac{5}{2} (E_j)_o G_j$$

$$= \frac{1}{2} (5E_j f_j + G_j E'_j)_o + 2C_{\varepsilon 1} C_\mu (f_j'^2 K_j)_o - C_\mu C_{\varepsilon 2} (\frac{E_j K_o}{D_j})_o \tag{3.50c}$$

$$\frac{P_{j-1}}{\Delta\xi} - \frac{P_j}{\Delta\xi} + \frac{1}{2} (f_{j-1} + f_j)_o f_{j-1} + \frac{1}{2} (f_{j-1} + f_j)_o f_j$$

$$= \frac{1}{2} (f^2_{j-1} + f^2_j)_o \tag{3.50d}$$

$$\frac{G_{j-1}}{\Delta\xi} - \frac{G_j}{\Delta\xi} + \frac{1}{2} f_{j-1} + f_j = o \tag{3.50e}$$

The preceeding system of difference equations (3.50) can be written in the following matrix form:

$$\begin{bmatrix} a_{11} & 0 & 0 & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 & 0 \\ a_{31} & 0 & a_{33} & 0 & 0 \\ a_{41} & 0 & 0 & a_{44} & 0 \\ a_{51} & 0 & 0 & 0 & a_{55} \end{bmatrix} \begin{bmatrix} F_{j-1} \\ K_{j-1} \\ E_{j-1} \\ P_{j-1} \\ G_{j-1} \end{bmatrix} + \begin{bmatrix} b_{11} & 0 & 0 & 0 & b_{15} \\ b_{21} & b_{22} & b_{23} & 0 & b_{25} \\ b_{31} & b_{32} & b_{33} & 0 & b_{35} \\ b_{41} & 0 & 0 & b_{44} & 0 \\ b_{51} & 0 & 0 & 0 & b_{55} \end{bmatrix} \begin{bmatrix} F_j \\ K_j \\ E_j \\ P_j \\ G_j \end{bmatrix}$$

$$+ \begin{bmatrix} C_{11} & 0 & 0 & 0 & 0 \\ C_{21} & C_{22} & 0 & 0 & 0 \\ C_{31} & 0 & C_{33} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} F_{j+1} \\ K_{j+1} \\ E_{j+1} \\ P_{j+1} \\ G_{j+1} \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \\ 0 \end{bmatrix} \tag{3.51}$$

where the a's, b's and c's are the coefficients in the difference equations (3.50) and the d's are on the right hand side of the equations. At the inner boundary (ξ=0) the system of equations in (3.51) will be replaced by the boundary conditions (3.45). At the outer edge the functions F, K and E go to zero, and G and P will be evaluated using equations (3.50d and e).

3.8 Similarity Solution

The system of equations (3.51) is of the same form that is given by equation (2) in Appendix F. Unlike the conventional numerical method, the numerical scheme which has been introduced in Appendix F solves for the unknowns at each grid point simultaneously. This method eliminates some of the errors arising from the coupling of the equations of motion.

To start the numerical solution we have to guess some suitable profiles. From this first guess the coefficients in the difference equations and the source terms will be evaluated. Then using the scheme mentioned above (Appendix F) a new profile will be calculated. With the new profiles, the coefficients and source terms will be updated and another iteration will take place. This procedure will be repeated until the differences between successive solutions reach a certain fixed tolerance, signifying convergence to the desired solution.

The initial profiles are obtained from the eddy viscosity solution which is given in Appendix B. The mean velocity profiles are given by the exact expressions (B-14) and(B-26). The parameter C in these equations is selected based on experimental data. The kinetic energy of turbulence and the dissipation rate will be estimated based on the observed values of K E and $\overline{uv}$ (See Appendix C).

Several attempts have been made to predict the flow field of the plane and round jets using the initial profiles described above,but we could not find a reasonable solution with the set of constants that had been originally proposed by Jones and Launder (1972) and given in Table 3.1. It has been observed (present study) that the k-ε model was not so sensitive to the constant in the diffusion term as it was for the constant in the production/destruction terms in the ε equation. This behavior has been found not only in the axisymmetric jet but also in the plane jet as well.

The constants associated with the k-ε model appear not to be universal since they differ from one flow to the other. In particular, a set of constants that works well in plane free shear flows will not do as well in the axisymmetric case. Over the last decay several analyses have been made to establish a set of model constants that agree well with experimental data. For example, Launder et al. (1973) have done extensive studies of turbulent free shear flows and reevaluated the model constants (see Table 3.2). Hoffman (1975) examined the constants in the diffusion terms of the k- and ε- equations for channel flows. Pope (1978) analyzed the plane and round jet,and he added an extra term in the dissipation equation for the round jet case to account for the vortex stretching effect.

Hassid (1979) solved the k-ε model for momentumless wakes and suggested another set of constants. Hanjalic and Launder (1980) proposed a modified dissipation equation and added the second order terms to the production in the k- and ε- equations. They predicted the plane and axisymmetric turbulent jets with their improved model and concluded that further improvement of the model will widen its application to a large range of shear flows. Table (3.2) shows a comparison of the recently proposed model constants for the k-ε model.

Reference	Flow	C_μ	$C_{\varepsilon 1}$	$C_{\varepsilon 2}$	σ_k	σ_ε
Jones and Launder (1972)	Boundary layer	.09	1.55	2.0	1.0	1.3
Launder et al. (1973)	Free shear flows	.09	1.44	1.92	1.0	1.3
Hoffman (1973)	Channel flow	.09	1.81	2.0	2.0	3.0
Pope (1978)	Plane jet & round jet	.09	1.45	1.90	1.0	1.3
Hassid (1979)	Momentumless wake	.1667	1.44	1.92	1.0	2.0
Hanjalic and Launder (1980)	Free Shear Flows	.09	1.44	1.90	1.0	1.3
Present study	Plane jet	.09	1.45	2.0	1.0	2.0
	Round jet	.09	1.55	2.0	1.0	2.0

Table 3.2 A Comparison of the Proposed Model Constants in the k-ε model.

3.9 Proposed Model Constant

At the outer edge of the free shear flows such as the turbulent jets and wakes one may expect that the diffusion in the k- and ε- equations is balanced by the convection and that the dissipation is identically balanced by the production. Hence if we evaluate equations (3.16) and (3.17) near the outer edge of the flow we have:

$$y^i U\frac{\partial k}{\partial x} + y^i V\frac{\partial k}{\partial y} = \frac{\partial}{\partial y}\left(y^i \frac{C_\mu}{\sigma_k}\frac{k^2}{\varepsilon}\frac{\partial k}{\partial y}\right) \tag{3.52a}$$

$$y^i U\frac{\partial \varepsilon}{\partial x} + y^i V\frac{\partial \varepsilon}{\partial y} = \frac{\partial}{\partial y}\left(y^i \frac{C_\mu}{\sigma_\varepsilon}\frac{k^2}{\varepsilon}\frac{\partial \varepsilon}{\partial y}\right) \tag{3.52b}$$

Further, at the outer edge ($y \to \infty$), the lateral component of the mean velocity V approaches a constant value so that:

$$V \cong \text{const} \tag{3.53a}$$

$$V >> U \tag{3.53b}$$

and

$$\frac{\partial}{\partial y} >> \frac{\partial}{\partial x} \tag{3.53c}$$

Hence the equations (3.52) can be written as:

$$\frac{\partial}{\partial y}(y^i V k) = \frac{\partial}{\partial y}\left(y^i \frac{C_\mu}{\sigma_k}\frac{k^2}{\varepsilon}\frac{\partial k}{\partial y}\right) \tag{3.54a}$$

$$\frac{\partial}{\partial y}(y^i V \varepsilon) = \frac{\partial}{\partial y}\left(y^i \frac{C_\mu}{\sigma_\varepsilon}\frac{k^2}{\varepsilon}\frac{\partial \varepsilon}{\partial y}\right) \tag{3.54b}$$

Integration with respect to y leads to:

$$\frac{C_\mu}{\sigma_k}\frac{k^2}{\varepsilon}\frac{\partial k}{\partial y} - kV = 0 \tag{3.55a}$$

$$\frac{C_\mu}{\sigma_\varepsilon}\frac{k^2}{\varepsilon}\frac{\partial \varepsilon}{\partial y} - \varepsilon V = 0 \tag{3.55b}$$

where it can be shown that the integration constants are zero in this case.

Dividing equation (3.55a) by k, equation (3.55b) by ε, and subtracting yields:

$$C_\mu \frac{k^2}{\varepsilon}\left(\frac{1}{K\sigma_k}\frac{\partial}{\partial y}K - \frac{1}{\varepsilon}\frac{1}{\sigma_\varepsilon}\frac{\partial\varepsilon}{\partial y}\right) = o \tag{3.56}$$

Direct integration gives:

$$\frac{k^{\sigma_\varepsilon/\sigma_k}}{\varepsilon} = \text{const} \tag{3.57}$$

Now from equation (3.57 and the viscosity hypothesis (3.14) we must have:

$$\sigma_\varepsilon = 2\,\sigma_k \tag{3.58}$$

In the present study we have found that unless $\sigma_\varepsilon = 2\,\sigma_k$ the eddy viscosity (k^2/ε) will not take asymptotic value at the outer edge of the flow but increases instead. Therefore if we chose $\sigma_k = 1.0$ then $\sigma_\varepsilon = 2.0$. The best fit constants for the similarity solution are included in Table 3.2.

3.10 Results and Discussion of the k-ε Model

Several computer runs were made for different values of the model constants presented in Table 3.2. The results were analyzed and compared with the best available experimental data.

The constant σ_k in the diffusion term of the k-equation is, in fact, arbitrary and the choice of $\sigma_k = 1.0$ seems to be the simplest choice. However, this fixes σ_ε (Section 3-9) and transfers any adjustment of the diffusion coefficients to C_μ. On the other hand, $C_{\varepsilon 2}$ was evaluated from the data of decaying isotropic turbulence (Section 2.6) where the value $C_{\varepsilon 2} = 2$ was the asymptotic value of $C_{\varepsilon 2}$ for a large turbulent Reynolds number. In other words we may say that $C_\mu = .09$ and $C_{\varepsilon 1} = 1.45$ or $C_{\varepsilon 1} = 1.55$ are obtained by computer optimization for fixed values of σ_k, σ_ε and $C_{\varepsilon 2}$. Changing C_μ by a few percent will not effect the overall result as much as it did when the change was made in $C_{\varepsilon 1}$. For example, for $C_{\varepsilon 1} = 1.45$ an excellent agreement with the data was achieved in the plane jet; for the same value of $C_{\varepsilon 1}$, the velocity profile for the round jet was too narrow (i.e., $\frac{d\ell}{dx} = .064$ instead of $\frac{d\ell}{dx} = .086 - .09$). The choice of $C_{\varepsilon 2} = 1.55$ in the axisymmetric jet led however to a better result.

In previous theoretical work (eg. Taulbee and Lumley 1980) the decay of the centerline mean velocity and the spreading rate have been given great attention. The importance of these particular quantities - spreading rate and the decay rate of mean velocity - will become apparent when we discuss the momentum conservation in a later chapter. Unlike the routines of Taulbee and Lumley (1980) and others who integrated downstream until self-preservation was reached, the similarity solution presented here was obtained for the self-preservation region. Hence the spreading rate parameter in the integration routine will be equivalent to the width parameter of the mean velocity profile in the similarity solution.

A convenient measure for the spreading rate which is widely used in the literature is $\frac{d\ell}{dx}$, where the length scale ℓ is defined as the non-dimensional lateral distance at which the axial mean velocity is a half of its maximum. The constant C in the decay of the centerline mean velocity will be calculated from the momentum integral (equation (2.43)); that is,

$$\int_0^\infty f^2(\xi)\ \xi^i d\xi = \frac{(1-Q)}{2(i+1)^{i+1} C^2} \tag{3.52}$$

where Q is the fraction of turbulent contribution to the momentum transport and is given by:

$$Q = \int_0^\infty \left(g_1 - \frac{g_2+g_3}{2}\right) \xi^i d\xi \tag{3.53}$$

This turbulent contribution to the momentum integral will be neglected in the k-ε model because the model does not provide information about the normal stresses g_1, g_2 and g_3.

The left hand side of the equation (3.52) will be evaluated from the similarity solution; hence C can then be obtained by letting i=0 for the plane jet or i=1 for the round jet.

To display the quality of the agreement, the result obtained with the best-fit constants are presented together with the best available experimental data. For the axisymmetric jet the most comprehensive measurements are those of Wygnanski and Fiedler (1969). The more recent measurements for the round jet using a new method are those of Abbiss et al.(1975) and Rodi (1975). For the plane jet the comparisons of the results are made with reference to the experimental works of Bradbury (1965), Hekestad (1965), and Gutmark and Wygnanski (1975).

For meaningful comparisons, of course, it is very important that the measurements should have been performed at a cross section downstream where similarity prevails for both mean and fluctuating quantities. This requirement was not quite met in the case of Abbiss et al. where $x/d_0 = 30$; this was not far enough for the fluctuating quantities to achieve a self-preserving state. However, it has been observed (Wygnanski and Fiedler 1969 among others) that the mean velocity becomes self-similar at about $x/d_0 \leq 30$ so we can at least use the Abbiss et al. data for comparison of the calculated and measured mean velocity profiles.

Near the outer edge of the jets, the relative turbulence intensity is very high. As a consequence the measurements become increasingly unreliable toward the edges. Hence the discrepency between the calculated and measured profiles near the outer edge should not be attributed solely to the calculation.

The calculated mean and turbulent quantities for the plane and axisymmetric jet are plotted along with the measured profiles in the figures (3.3-3.14). The similarity solution for the plane jet predicts the mean velocity and kinetic energy profiles fairly well, as it can be seen in Figure (3.3) and (3.4). The shear stress ($\overline{uv}/U_m^2$) has been calculated using the eddy viscosity hypothesis (3.14) and (3.15). The result for the plane jet is shown in Figure (3.5) which displays good agreement with the data.

Figure (3.6) shows the energy budget across the jet, and it can be seen the terms in the energy equation are well balanced across the entire flow field. There are no accurate measurements available for the terms in the dissipation rate equation; however, from the calculated results, Figure (3.7) shows that the ε-equation is fairly balanced. From Figure (3.8) it can be seen that the calculated eddy viscosity ($\nu_t \sim k^2/\varepsilon$) is constant over most of the cross-section, and more importantly the ratio k^2/ε is well behaved at the outer edge of flow as both k and $\varepsilon \to o$ when $\xi \to \infty$.

For the round jet, Figures (3.9) and (3.10) show that the predicted mean velocity and, in turn, the shear stress display fairly good agreement with the data. The shear stress for the round jet was obtained in a similar way as that for the plane jet. The kinetic energy profile as shown in Figure (3.11) agrees with the data for $\xi > .06$, but it is off by nearly 18% near the axis ($\xi=0$). The reason for the energy loss near the axis of symmetry is due to the high dissipation rate "ε" as it can be seen in the energy balance figure (3.12). Figure (3.12) shows, however, that the kinetic energy equation is well in balance for $\xi > .06$.

The main errors in the prediction of the axisymmetric profiles are attributed to the dissipation rate transport equation. From the balance of ε-equation (Figure 3.18) we can see that the equation is not nearly in balance near the axis, and that the destruction term ($C_{\varepsilon 2}\varepsilon^2/k$) is too high near the center line and too low at the outer edge. The production term ($C_{\varepsilon 1}\nu_t \frac{\partial U}{\partial y}$)

seems to be too low to account for the destruction term near the centerline. However, the production/destruction terms approach the same order of magnitude for $\xi \geq .065$.

Nothing much can be said about the diffusion term because there is not enough information about the diffusion of ε; and as stated earlier, changing the constant C_μ in the diffusion will not change the final result because of the change in the constants for the production/destruction terms, $C_{\varepsilon 1}$ and $C_{\varepsilon 2}$.

The kinetic energy profile calculated by Hanjalic and Launder (1980) resulting from their improved k-ε model also shows a loss of roughly 20% of the energy near the axis in the axisymmetric jet. However, they did not show any energy balance for either the plane nor the axisymmetric jet.

Figure (3.14) shows the eddy viscosity ($\nu_t \sim k^2/\varepsilon$) for the round jet across the flow. The shape of k^2/ε seems to be unaffected by the energy loss near the center line and it is also behaving very nicely as $\xi \to \infty$. However, unlike the two-dimensional case the ratio k^2/ε decreases very slowly as increases at the edge of the jet.

Table (3.3) shows a comparison with the data of the spreading rate and the centerline mean velocity decay rate for both plane and axisymmetric jets. The spreading rate for both cases displays good agreement with the data. The values of the constant C seem to be overestimated as compared with the measured values in both cases, i.e., the plane and round jet. This over-estimate, however, is not an error in the calculation and it is only due to neglecting the contribution of the turbulence to the momentum transport. However, we will hold off further discussion of the constant C and the momentum conservation until we predict the jet flows using the Reynolds stress closure which solves directly for all the non-zero components at the Reynolds stress.

3.11 Concluding Remarks

From the foregoing results we may conclude that the k-ε model with the present set of constants (Section 3.7) has predicted the behavior of the turbulent plane jet in the similarity region within the experimental accuracy. The same can not be said, however, about the result of the axisymmetric jet, since the solution did not converge properly as it did for the plane case. The result presented here for the round jet is the best that can be obtained for the set of constants given in Table 3.2.

The lack of universality of the model constants and the error in the energy profiles support the thesis of section (2.6) and (2.7) that $C_{\varepsilon 1}$ and $C_{\varepsilon 2}$ might not be constants. In order for $C_{\varepsilon 1}$ and $C_{\varepsilon 2}$ to be universal they must predict the flow for both cases, plane and round jet. So far there have been no reliable measurements of the terms in the ε-transport equation, thus proper choices of the functional dependence of $C_{\varepsilon 1}$ and $C_{\varepsilon 2}$ is difficult at the present time. The overall results obtained with the k-ε model are indeed encouragement to further improve the model.

	Reference	$\frac{d\ell}{dx}$	C	x/D
Plane Jet	Present study, k-ε similarity solution	.1106	2.462	
	Bradbury (1965)	.109	2.4	14-70
	Heskestad	.11		47-155
	Gutmark and Wygnanski (1976)	.102	2.306	10-150
Round Jet	Present study, k-ε similarity solution	.087	6.4	
	Wygnanski and Fiedler (1969)	.086	5.0*	20-98
	Rodi (1975)	.086	6.0	20-75
	Abbiss et al.	.089-.1	5.5	20-30

Table 3.3 Spreading and Decay Rate Constants for Plane and Axisymmetric Jet.

*The values of C have been obtained from the graphs which are given by the authors.

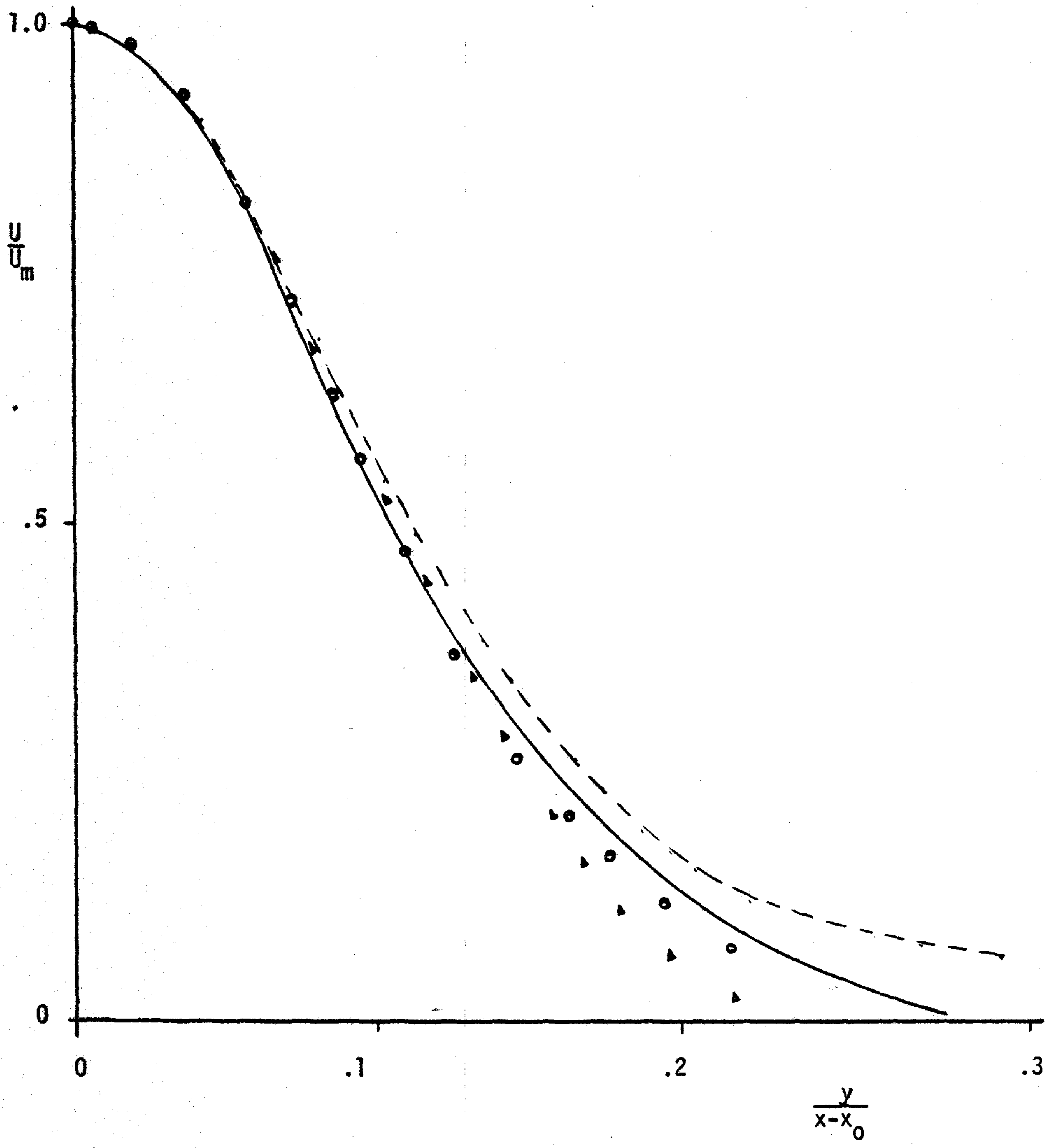

Figure 3.3 Mean Velocity Profile in Plane Jet

(——) Similarity solution
(O) Gutmark and Wygnanski (1976)
(Δ) Bradbury (1965)
(---) Heskastad (1965)

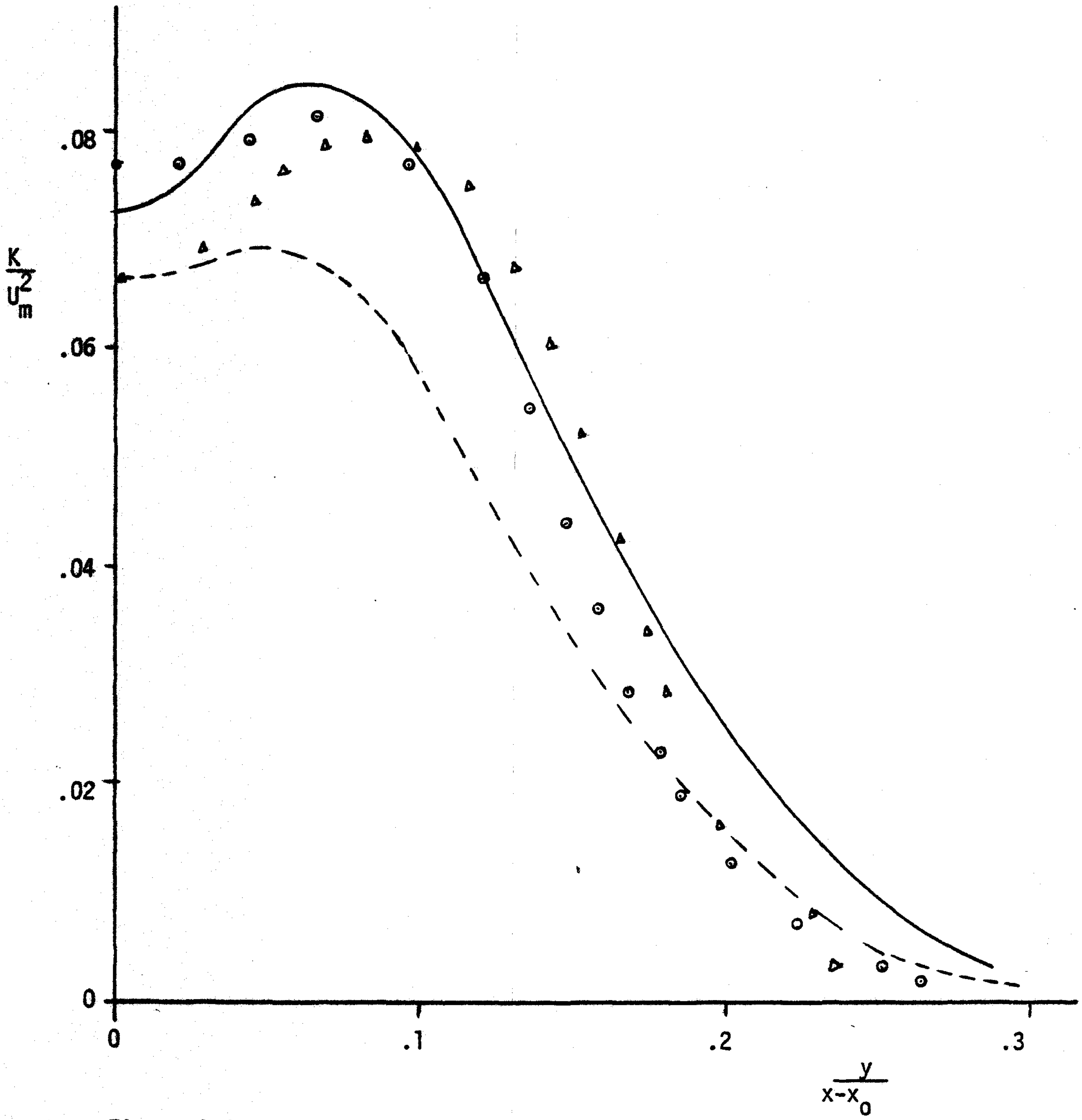

Figure 3.4 Turbulence Kinetic Energy in Plane Jet. Notations as in Figure 3.2.

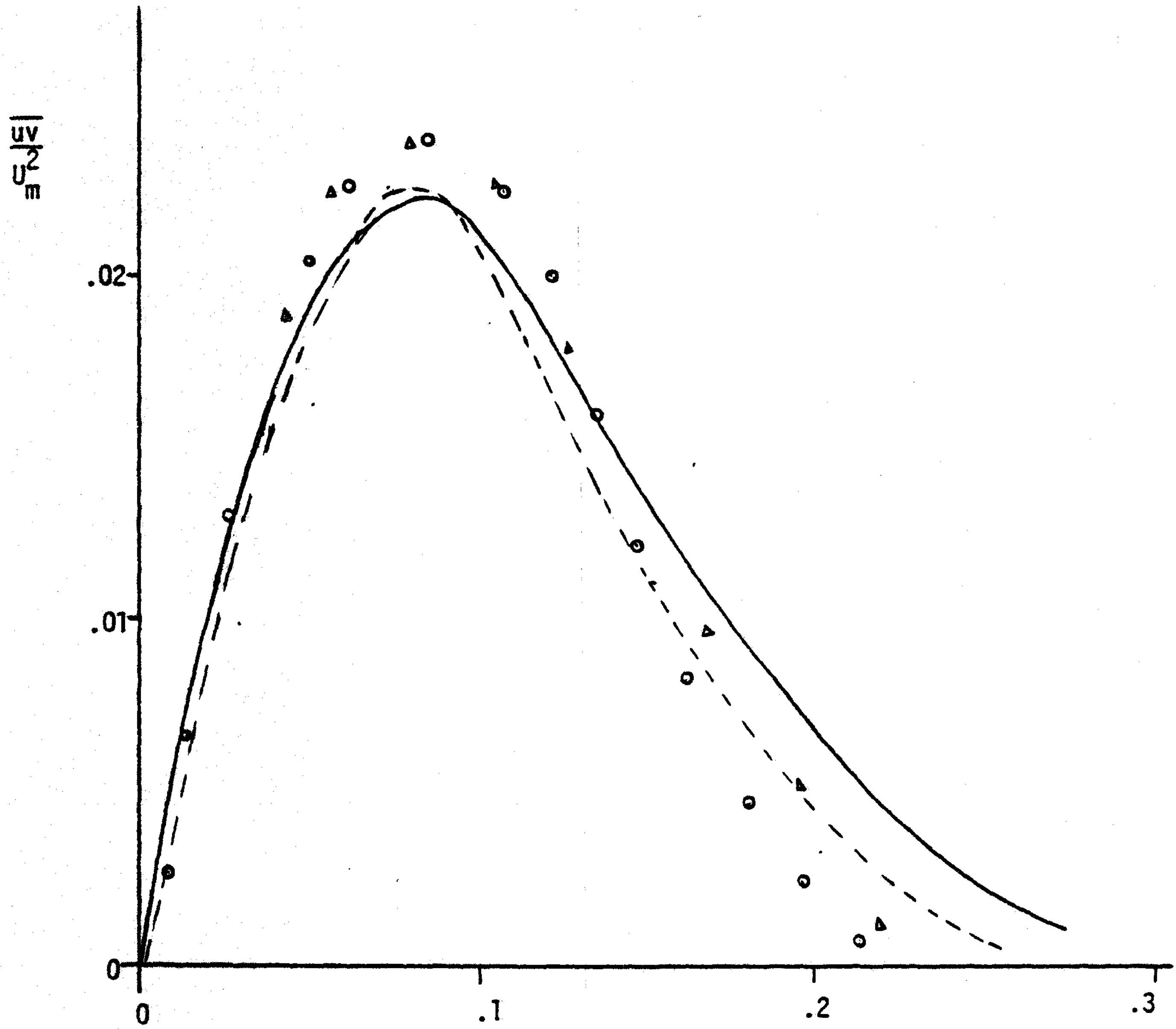

Figure 3.5 Shear Stress Distribution in Plane Jet.

Notations as in Figure 3.2.

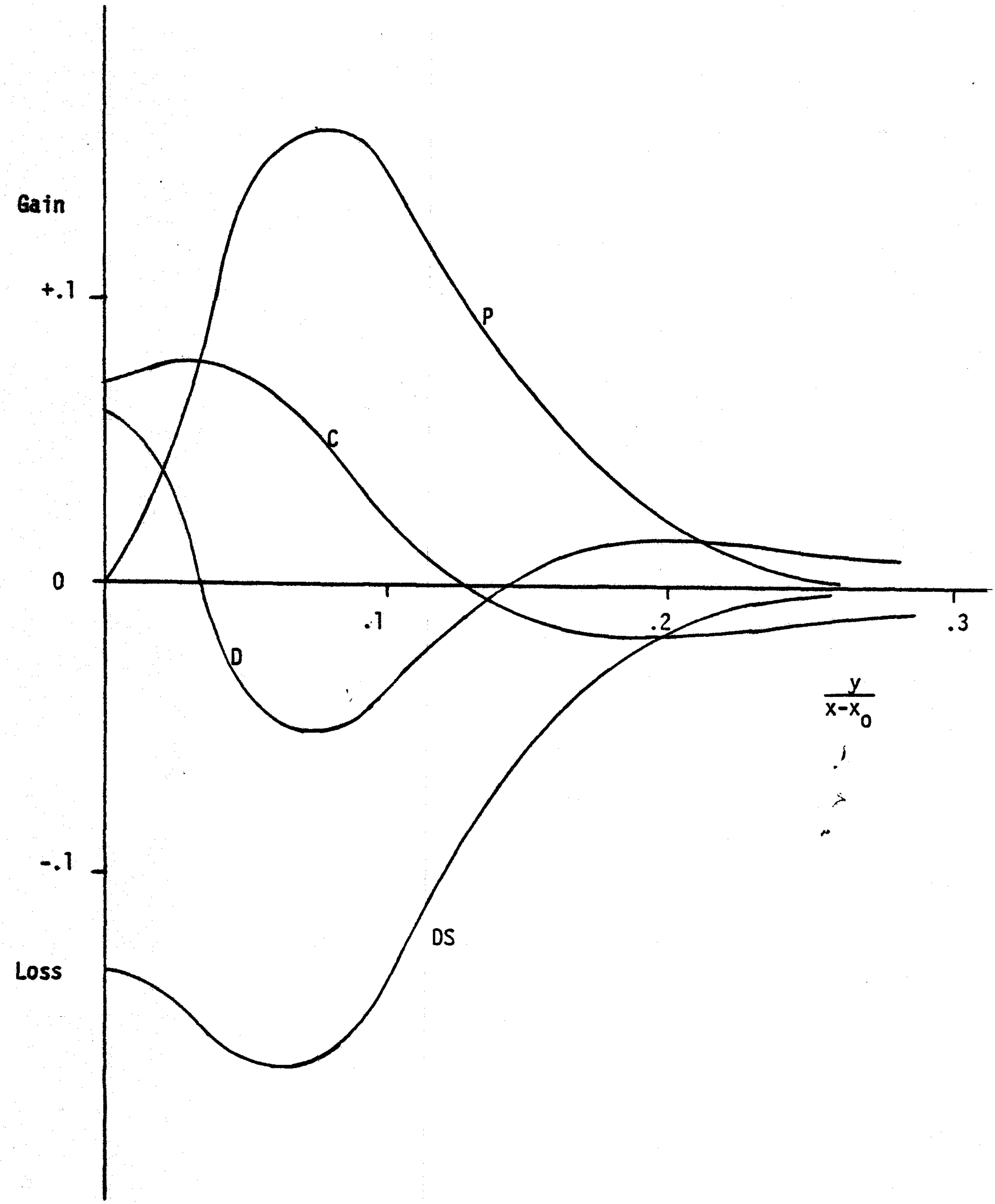

Figure 3.6 Calculated Turbulence Energy Balance Across the Flow in Plane Jet where:

C = convection, D = diffusion, P = production, DS = dissipation

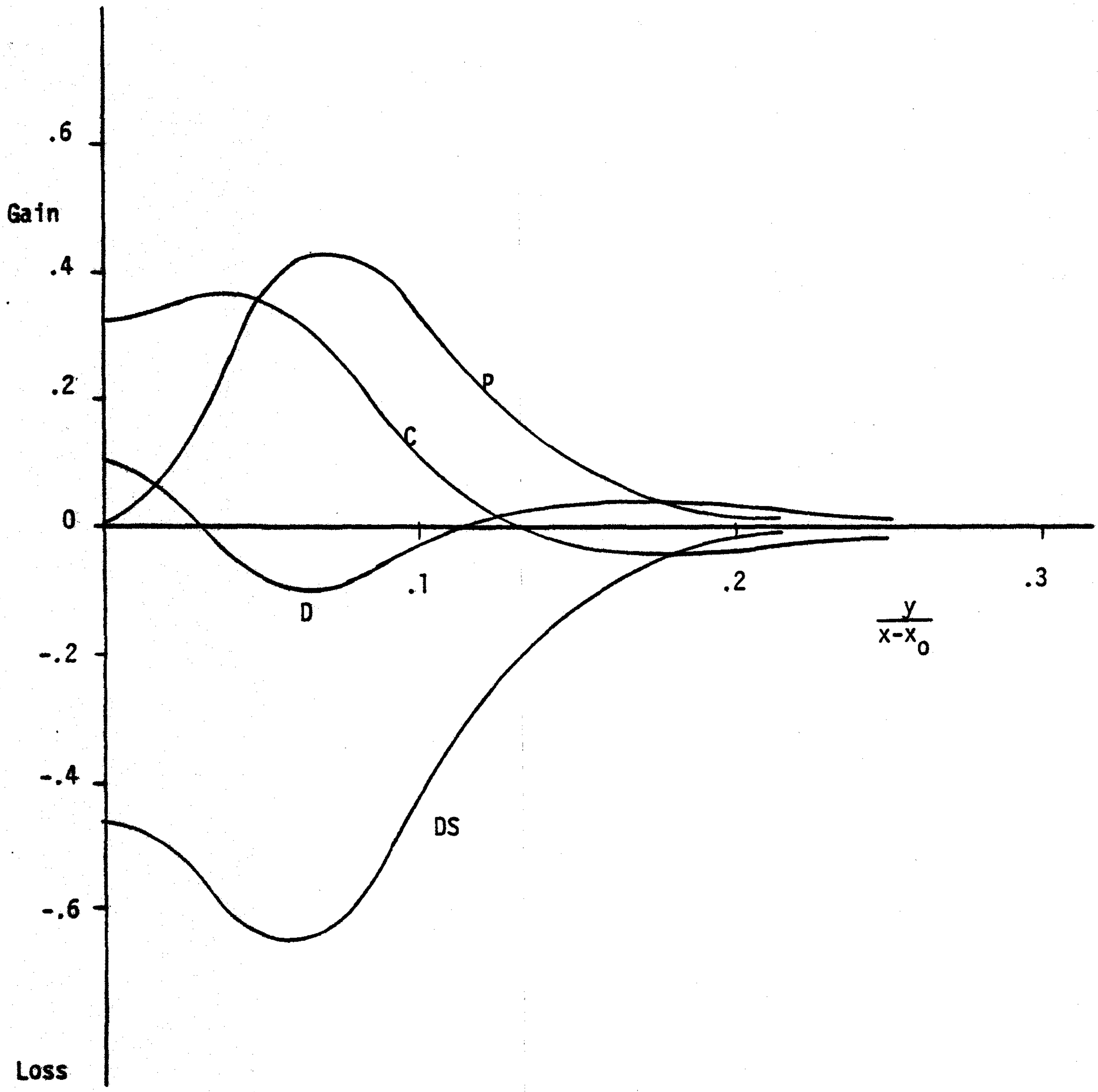

Figure 3.7 Calculated Dissipation Rate "ε" Balance Across Self-Preserving Plane Jet where:

C = convection, D = diffusion, P = production, DS - Destruction of ε.

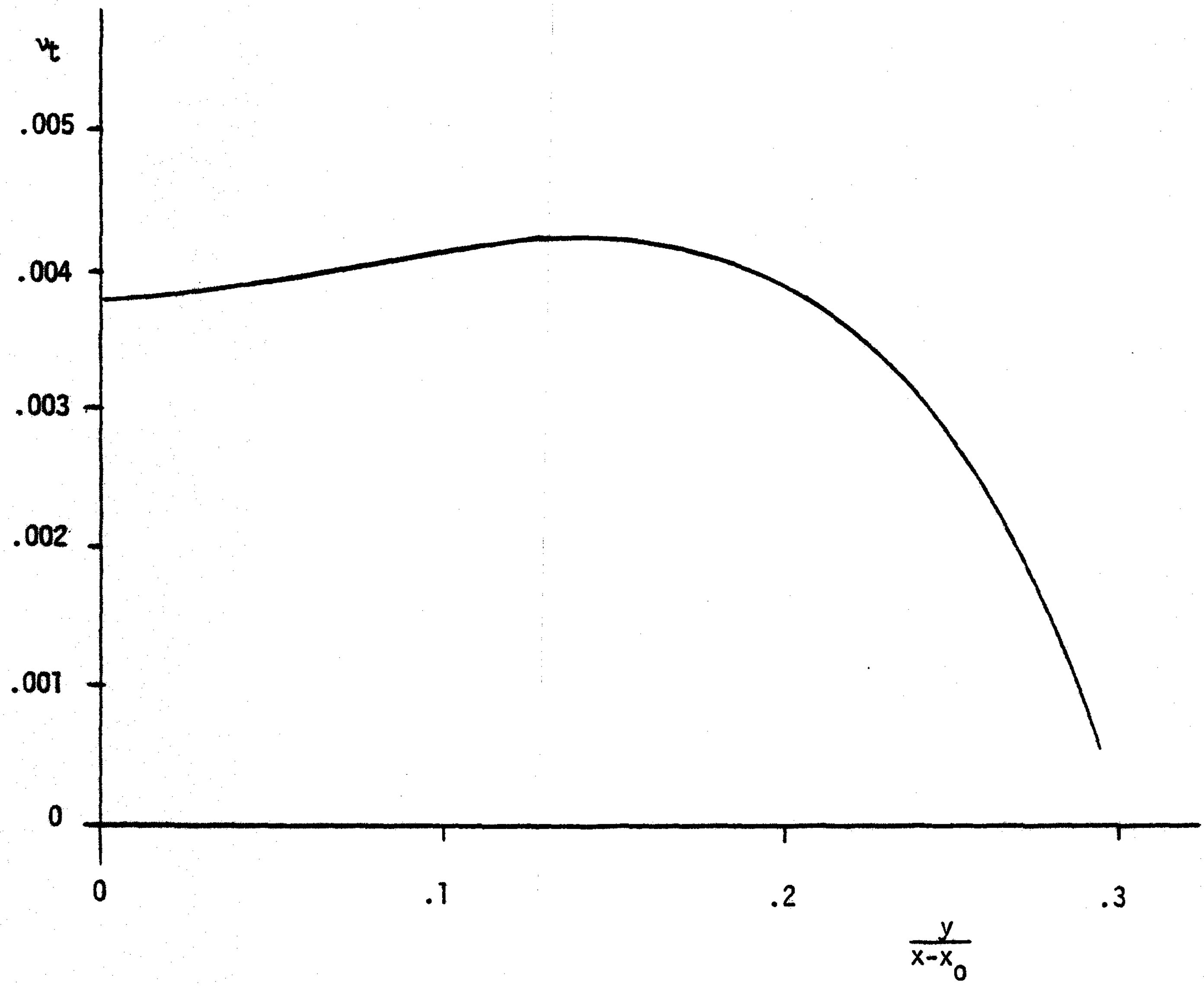

Figure 3.8 Calculated Turbulence "eddy viscosity" Across the Flow in Plane Jet ($\nu_t = C_\nu k^2/\varepsilon$).

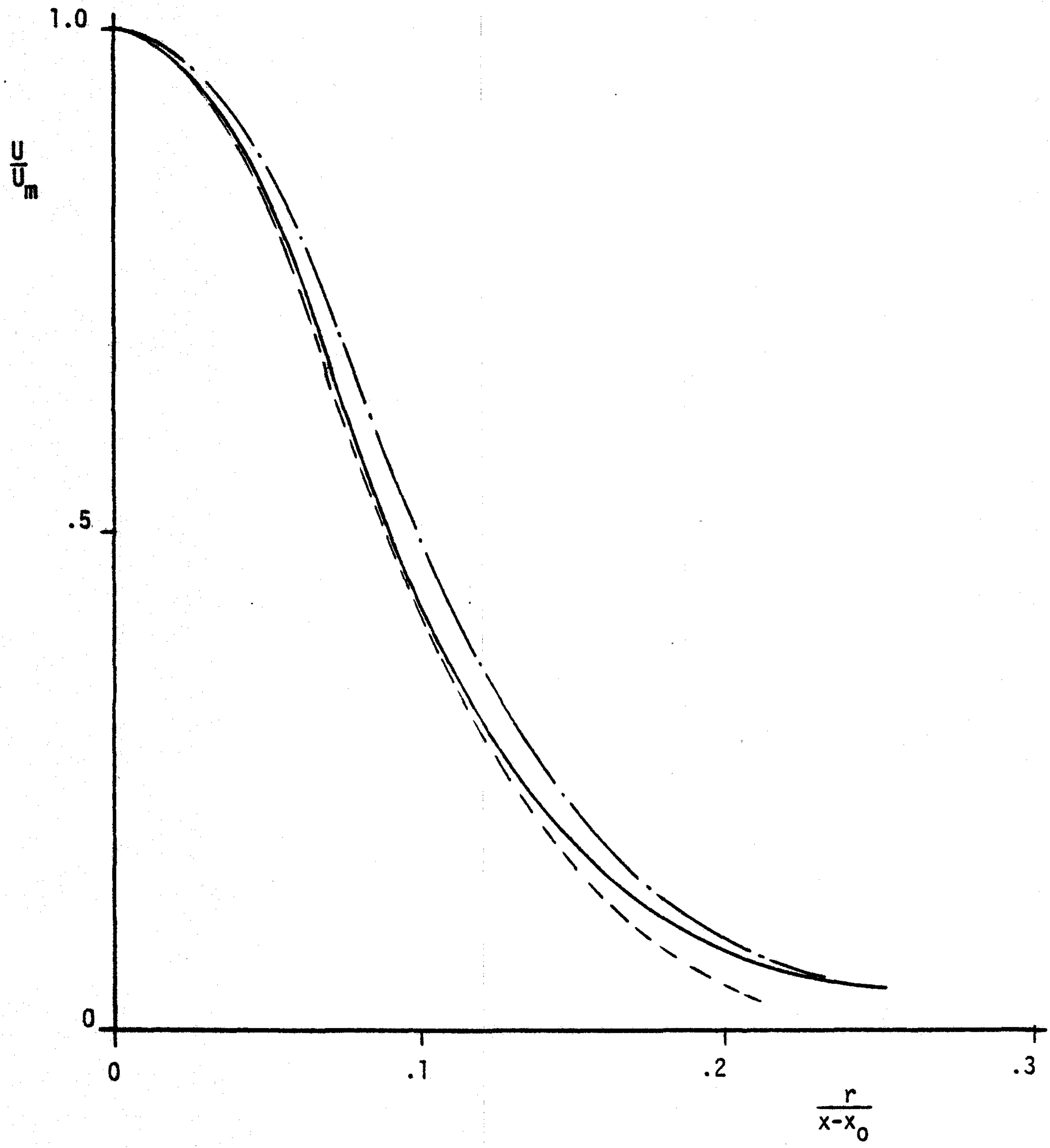

Figure 3.9 Mean Velocity Profile in Axisymmetric Self-preserving Turbulent Jet

(——) Similarity Solution (k-ε model)
(-·-) Abbiss et al. (Pulsed wire) (1975)
(---) Wygnanski & Fiedler (1969).

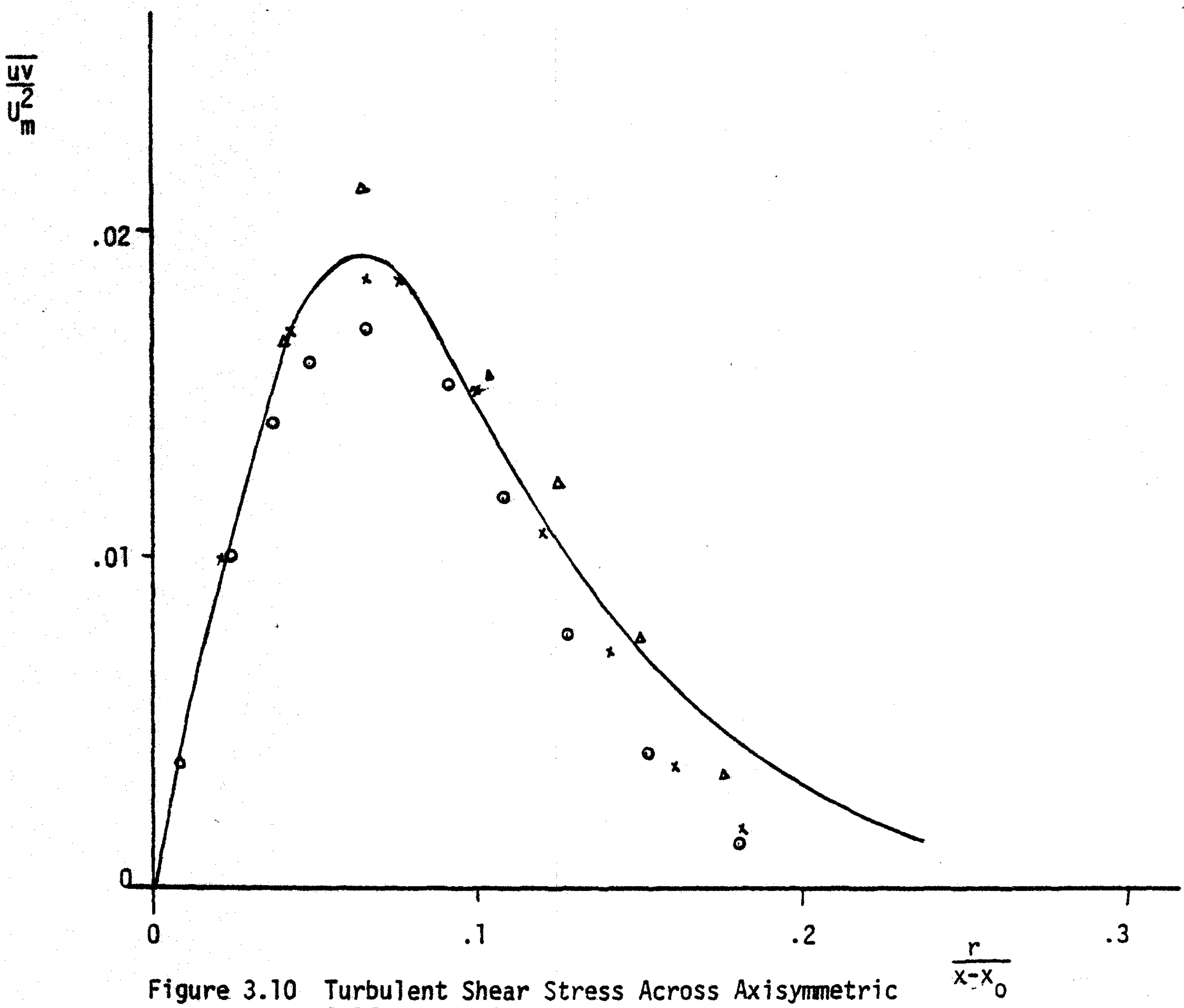

Figure 3.10 Turbulent Shear Stress Across Axisymmetric Self-preserving Jet.

(——) Similarity Solution (k-ε model)
(o) Wygnanski and Fiedler (1969)
(Δ) Rodi (1975)
(x) Abbiss et al. (Pulsed wire) (1975)

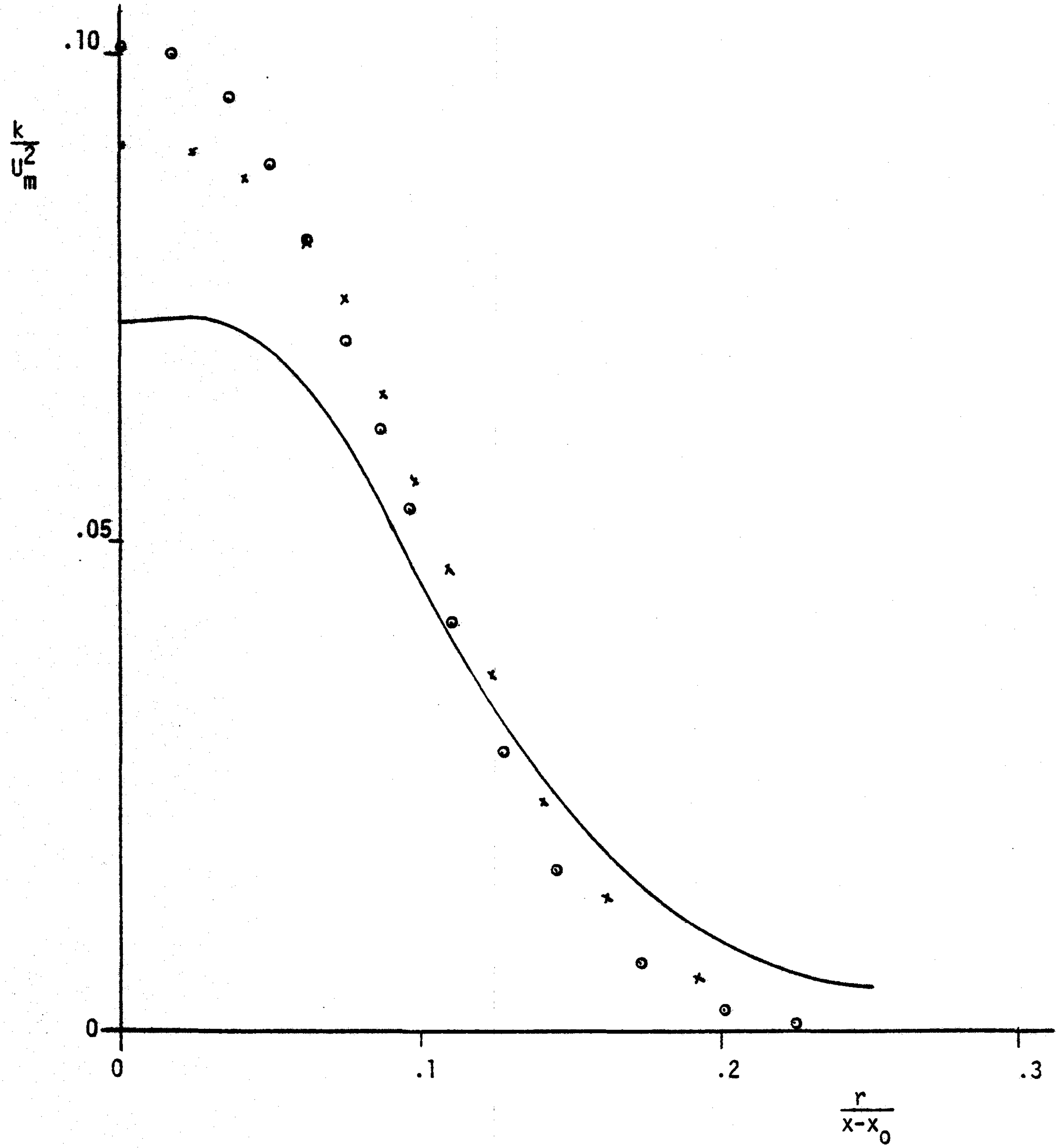

Figure 3.11 Turbulence Kinetic Energy Profile in Self-preserving Axisymmetric Turbulent Jet.

(——) Similarity Solution (k-ε model)
(o) Wygnanski and Fiedler (1969).
(x) Rodi (1975).

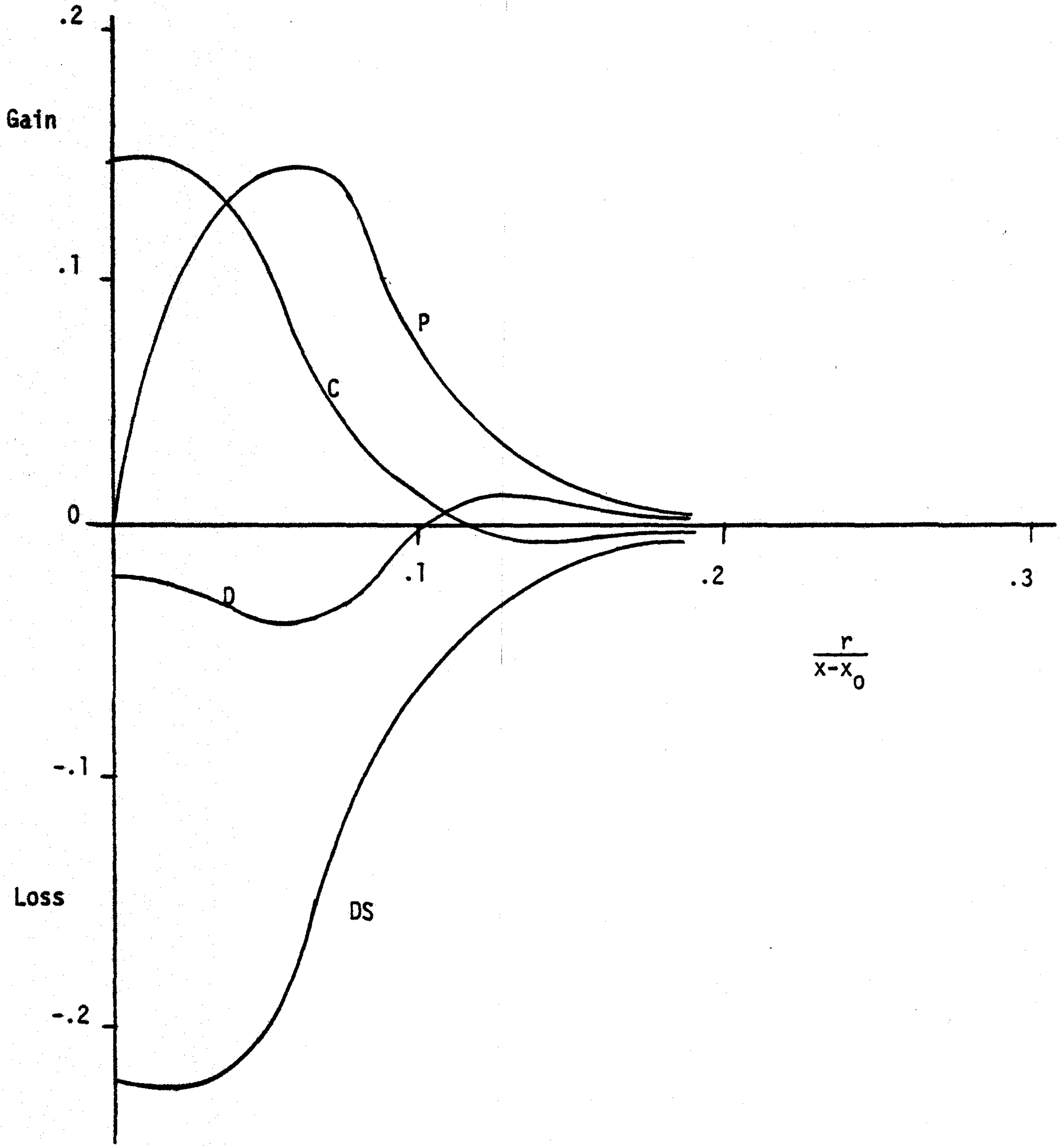

Figure 3.12 Calculated Kinetic Energy Balance Across Self-preserving Axisymmetric Turbulent Jet, where:

C = convection, D = diffusion, P = production, DS = dissipation

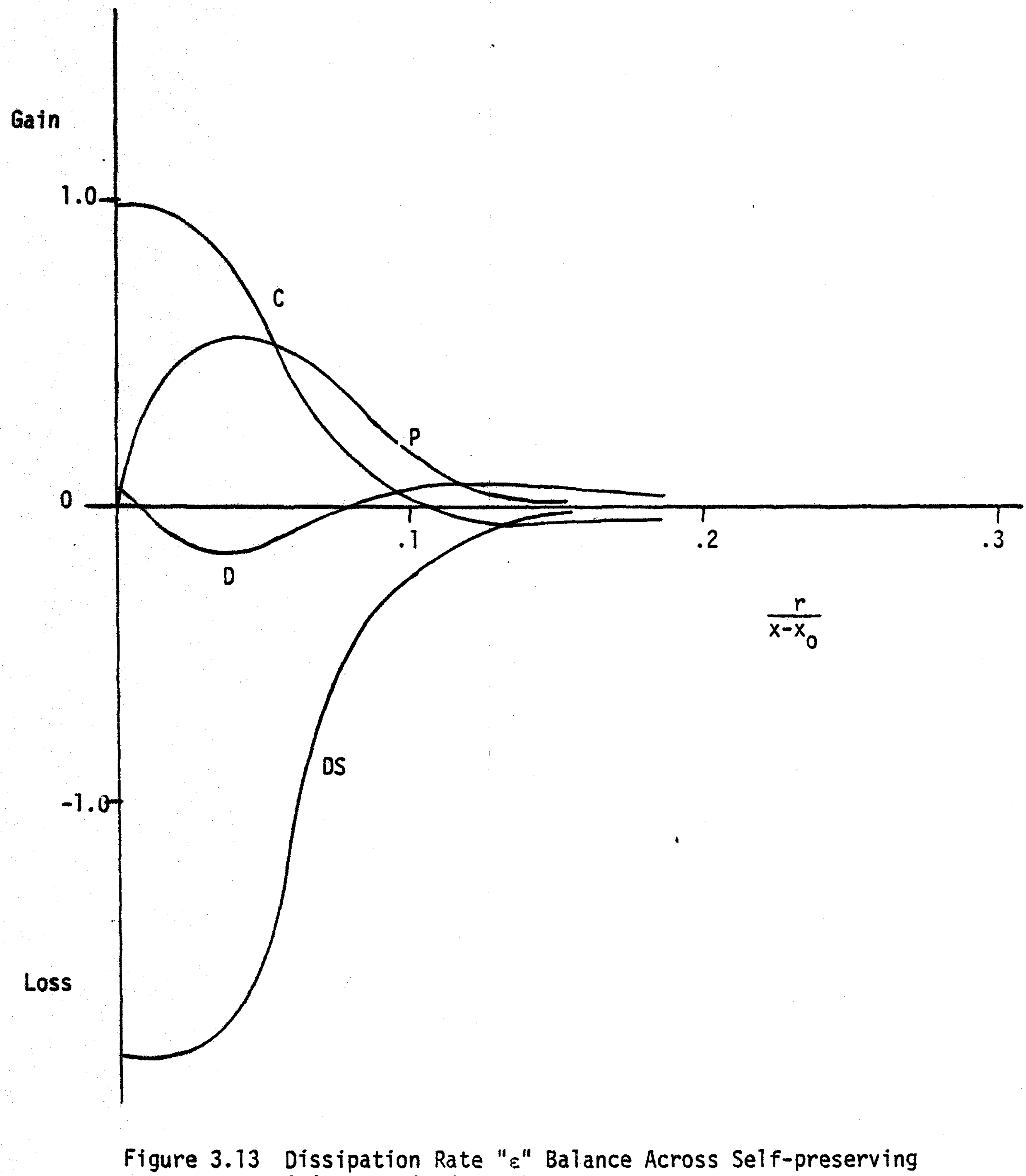

Figure 3.13 Dissipation Rate "ε" Balance Across Self-preserving Axisymmetric Jet, where:

C = convection, D = diffusion, P = production, DS = destruction of ε.

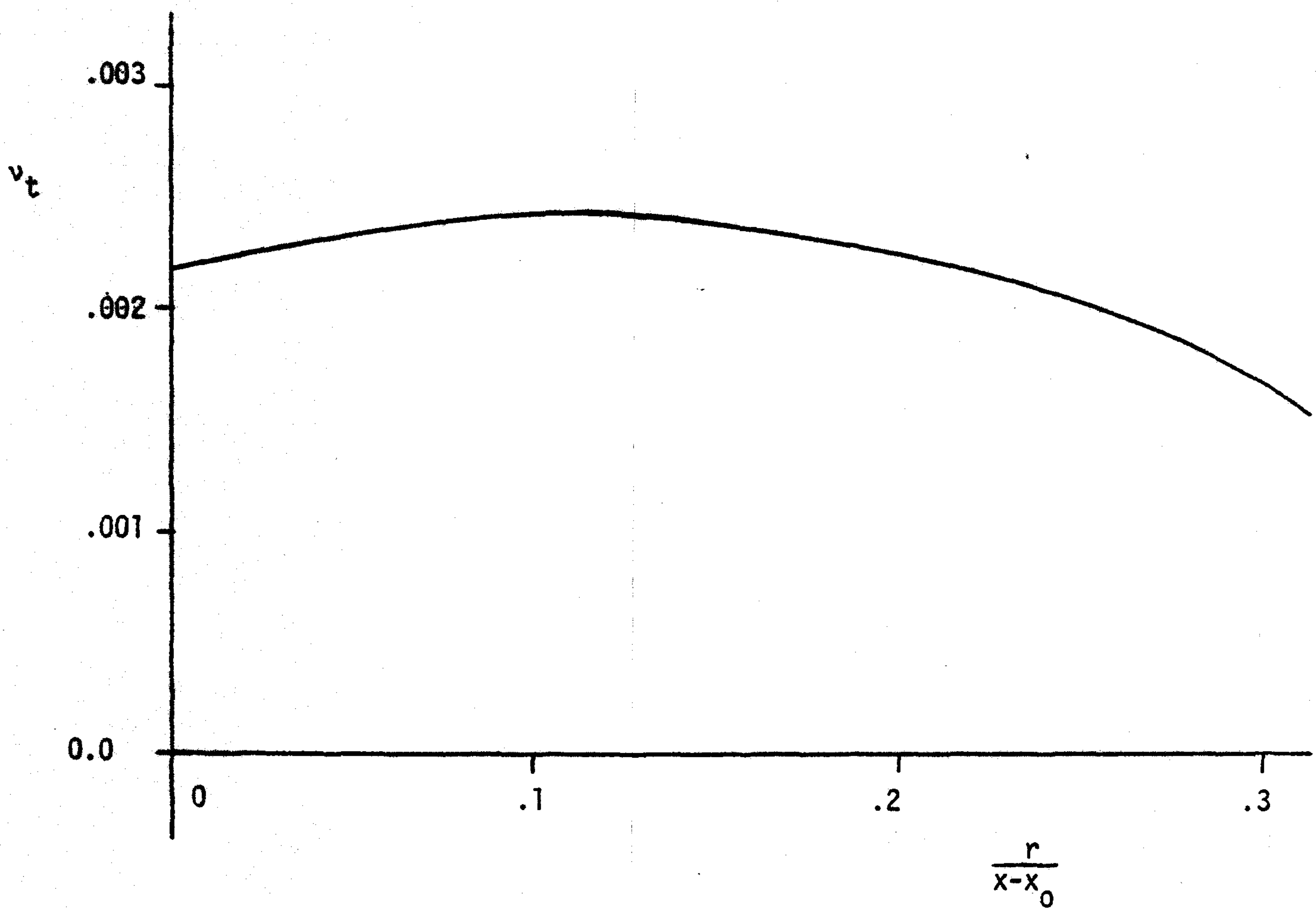

Figure 3.14 Calculated Turbulent "eddy viscosity" Across Self-preserving Axisymmetric Jet.

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CHAPTER 4

Application of the Reynolds Stress Model

4.1 Introduction

In Chapter 3, the plane and round turbulent free jets have been predicted using the k-ε model closure. As discussed earlier, a single set of constants could not produce good results for both the plane and axisymmetric jets. Further, the kinetic energy in the round jet was 18% lower than the data. Our primary aim here is to reexamine the plane and round jet flows using the Reynolds stress model and compare the results with available experimental data. Here, unlike in the k-ε model, the parameter in the destruction term of ε-equation (ψ_0) which dictates the rate of the kinetic energy decay is a function of the state of turbulence. Hence with ψ_1 = constant, ψ_0 will be adjusted as a function of the turbulence Reynolds number and the first invariant of the anisotropy tensor. Also the diffusion terms in the Reynolds stress model are controlled by the parameter C_1 which will be part of the calculation as a function of Re, II and III. It is expected that the parameters C_1 and ψ_1 with the fixed constants ψ_1 and c are more likely to be universal than the pure constants in the k-ε model.

The equations of motion for the mean flow are given by equations (2.1) and (2.2) and they are discussed in Appendix A for both plane and axisymmetric jet. The stress equations are obtained from equation (2.67). In Appendix D the Reynolds stress equations for each stress component have been written in a cartesian coordinate system (plane jet). In Appendix E, the equations for the Reynolds stress (2.67) has been transformed to curvilinear form and the equations for the axisymmetric case have been written for x, r and θ components. In Section (4.2) the equations for the stress components and the dissipation rate have been arranged for a general free shear flow, so that the extra terms arising in this axisymmetric case can be eliminated and the problem becomes two-dimensional. The equations for the mean and turbulence qualities are then solved simultaneously for the self-preserving jet. The centerline values of the stress components and the dissipation have been updated after each iteration. The final solutions are compared with the existing experimental data.

4.2 The Reynolds Stress Equations

The governing equations for the kinematic Reynolds stresses for an incompressible and isothermal turbulent flow are given in Appendix D for the plane jet and in Appendix E for the axisymmetric case. After some rearrangement the equations become:

$\overline{u^2}$-Equation

$$U \frac{\partial \overline{u^2}}{\partial x} + V \frac{\partial \overline{u^2}}{\partial y} = \frac{\partial}{\partial y} [C_o \frac{\overline{q^2}}{\varepsilon} \{(C_2+1)\ \overline{v^2}\ \frac{\partial \overline{u^2}}{\partial y} + 3C_2\ \overline{v^2}\ \frac{\partial \overline{v^2}}{\partial y} + C_2\ \overline{v^2}\ \frac{\partial \overline{w^2}}{\partial y}$$

$$+ 2(C_2+1)\ \overline{uv}\ \frac{\partial \overline{uv}}{\partial y}\}] + \frac{i}{y}\ C_o \frac{\overline{q^2}}{\varepsilon}\{C_2\ \overline{v^2}\ \frac{\partial \overline{u^2}}{\partial y} + 3(C_2+1)\ \overline{v^2}\ \frac{\partial \overline{v^2}}{\partial y}$$

$$+ (C_2-2)\ \overline{v^2}\ \frac{\partial \overline{w^2}}{\partial y} + 2C_2\ \overline{uv}\ \frac{\partial \overline{uv}}{\partial y}\} - C_1 \frac{\varepsilon}{\overline{q^2}}\ \overline{u^2} + \frac{1}{3} (C_1-2)\varepsilon$$

$$+ \frac{2}{3}(4C-1)\ \overline{uv}\ \frac{\partial U}{\partial y} + \underline{[-2\overline{u^2} + 4(\frac{1}{15} - cb_{11})\ \overline{q^2}]}\ \frac{\partial U}{\partial x}$$

$$\underline{+4[-\frac{1}{30} + c(b_{11}+b_{33})]\overline{q^2}}\ \frac{\partial V}{\partial y} + \underline{4[-\frac{1}{30} + c(b_{11}+b_{33})]\overline{q^2}\ \frac{V}{y}\ i}$$

$$+ \frac{i}{y} \frac{\partial}{\partial y} [2\ \frac{C_o C_2}{2}\ \overline{q^2}\ (\overline{v^2}\ \overline{w^2} - \overline{w^2}^2)] \qquad (4.1)$$

$\overline{v^2}$- Equation

$$U \frac{\partial \overline{v^2}}{\partial y} + V \frac{\partial \overline{v^2}}{\partial y} = \frac{\partial}{\partial y} [C_o \frac{\overline{q^2}}{\varepsilon} \{(C_2 - \frac{2}{3}) \overline{v^2} \frac{\partial \overline{u^2}}{\partial y} + 3(C_2 + \frac{3}{5}) \overline{v^2} \frac{\partial \overline{v^2}}{\partial y}$$

$$+ (C_2 - \frac{2}{5}) \overline{v^2} \frac{\partial \overline{w^2}}{\partial y} + 2(C_2 - \frac{2}{5}) \overline{uv} \frac{\partial \overline{uv}}{\partial y}\}] + \frac{i}{y} C_o \frac{\overline{q^2}}{\varepsilon} \{C_2 \overline{v^2} \frac{\partial \overline{u^2}}{\partial y}$$

$$+ 3(C_2+1) \overline{v^2} \frac{\partial \overline{v^2}}{\partial y} + (C_2-2) \overline{v^2} \frac{\partial \overline{w^2}}{\partial y} + 2C_2 \overline{uv} \frac{\partial \overline{uv}}{\partial y}\} - C_1 \frac{\varepsilon}{\overline{q^2}} \overline{v^2}$$

$$+ \frac{1}{3} (C_1-2)\varepsilon - \frac{4}{3} (1+5C) \overline{uv} \frac{\partial U}{\partial y} + \underline{4[-\frac{1}{30} + c(b_{11}+b_{22})]\overline{q^2} \frac{\partial U}{\partial x}}$$

$$+ \underline{[4(\frac{2}{30} - cb_{22})\overline{q^2} - 2\overline{v^2}] \frac{\partial V}{\partial y}} + \underline{4[-\frac{1}{30} + c(b_{22}+b_{33})]\overline{q^2} \frac{V}{y} i}$$

$$+ \frac{(i)}{y} \frac{\partial}{\partial y} [2C_o \frac{\overline{q^2}}{\varepsilon} (C_2 - \frac{2}{5})(\overline{v^2}\,\overline{w^2} - \overline{w^2}^2)] - \frac{(i)}{y^2} C_o \frac{\overline{q^2}}{\varepsilon} (\frac{16}{5})(\overline{\overline{v^2}\,\overline{w^2} - \overline{w^2}^2}) \tag{4.2}$$

$\overline{w^2}$- Equation

$$U \frac{\partial \overline{w^2}}{\partial x} + V \frac{\partial \overline{w^2}}{\partial y} = \frac{\partial}{\partial y} [C_o \frac{\overline{q^2}}{\varepsilon} \{C_2 \overline{v^2} \frac{\partial \overline{u^2}}{\partial y} + 3C_2 \overline{v^2} \frac{\partial \overline{v^2}}{\partial y} + (C_2+1) \overline{v^2} \frac{\partial \overline{w^2}}{\partial y}$$

$$+ 2C_2 \overline{uv} \frac{\partial \overline{uv}}{\partial y}\}] + \frac{i}{y} C_o \frac{\overline{q^2}}{\varepsilon} \{(C_2 - \frac{2}{5}) \overline{v^2} \frac{\partial \overline{u^2}}{\partial y} + 3(C_2 - \frac{2}{5}) \overline{v^2} \frac{\partial \overline{v^2}}{\partial y}$$

$$+ (C_2 + \frac{13}{5}) \overline{v^2} \frac{\partial \overline{w^2}}{\partial y} + 2(C_2 - \frac{2}{5}) \overline{uv} \frac{\partial \overline{uv}}{\partial y}\} - C_1 \frac{\varepsilon}{\overline{q^2}} \overline{w^2} + \frac{1}{3} (C_1-2)\varepsilon$$

$$+ 4\, c\, \overline{uv} \frac{\partial U}{\partial y} + \underline{4[-\frac{1}{30} + c(b_{11}+b_{33})] \overline{q^2} \frac{\partial U}{\partial x}} + \underline{4[-\frac{1}{30} + c(b_{22}+b_{33})]\overline{q^2} \frac{\partial V}{\partial y}}$$

$$+ \underline{[-2 \overline{w^2} + 4(\frac{2}{30} - cb_{33})\overline{q^2}] \frac{V}{y} i} + \frac{i}{y} \frac{\partial}{\partial y} [2\, C_o \frac{\overline{q^2}}{\varepsilon} (C_2+1)(\overline{\overline{v^2}\,\overline{w^2} - \overline{w^2}^2})]$$

$$+ \frac{i}{y^2} C_o \frac{\overline{q^2}}{\varepsilon} (\frac{16}{5})(\overline{\overline{v^2}\,\overline{w^2} - \overline{w^2}^2}) \tag{4.3}$$

<u>$\overline{uv}$- Equation</u>

$$U \frac{\partial \overline{uv}}{\partial y} + V\frac{\partial uv}{\partial y} = \frac{\partial}{\partial y} [C_o \frac{\overline{q^2}}{\varepsilon} \{- \frac{3}{5} \overline{uv} \frac{\partial \overline{u^2}}{\partial y} + \frac{4}{5} \overline{uv} \frac{\partial \overline{v^2}}{\partial y} - \frac{1}{5} \overline{uv} \frac{\partial w^2}{\partial y}$$

$$+ \frac{8}{5} \overline{v^2} \frac{\partial \overline{uv}}{\partial y}\}] + \frac{i}{y} C_o \frac{\overline{q^2}}{\varepsilon} \{\overline{uv} \frac{\partial \overline{v^2}}{\partial y} - \overline{uv} \frac{\partial \overline{w^2}}{\partial y} + 2 \overline{v^2} \frac{\partial \overline{uv}}{\partial y}\}$$

$$- C_1 \frac{\varepsilon}{q^2} \overline{uv} + [- \frac{7+50c}{15} \overline{u^2} + \frac{20c-2}{15} \overline{v^2} + \frac{2+20c}{10} \overline{w^2}] \frac{\partial U}{\partial y}$$

$$- (1+2c) \overline{uv} \frac{\partial U}{\partial x} - (1+2c) \overline{uv} \frac{\partial V}{\partial y} + 4c \overline{uv} \frac{V}{y} i - \frac{i}{y} \frac{\partial}{\partial y} \frac{2}{5} C_o \frac{\overline{q^2}}{\varepsilon} \overline{uv} \overline{w^2}$$

$$- \frac{i}{y^2} C_o \frac{\overline{q^2}}{\varepsilon} (\frac{8}{5}) \overline{uv} \overline{w^2} \qquad (4.4)$$

<u>ε - Equation</u>

$$U \frac{\partial \varepsilon}{\partial x} + \frac{\partial \varepsilon}{\partial y} = \frac{\partial}{\partial y} [\frac{9}{5(4C_1+9b+10)} \frac{\overline{q^2}}{\varepsilon} (\overline{v^2} + 2\frac{\overline{v^2}^2 + \overline{uv}^2}{\overline{q^2}}) \frac{\partial \varepsilon}{\partial y}]$$

$$+ \frac{i}{y} \{\frac{9}{5(4C_1+9b)+10} \frac{\overline{q^2}}{\varepsilon} (\overline{v^2} + 2\frac{\overline{v^2}^2 + \overline{uv}^2}{\overline{q^2}}) \frac{\partial \varepsilon}{\partial y} \}$$

$$- \psi_o \frac{\varepsilon}{q^2} \varepsilon - \psi_1 \varepsilon\, b_{12} \frac{\partial U}{\partial y} - \underline{\psi_1 \varepsilon\, b_{11} \frac{\partial U}{\partial x}}$$

$$- \underline{\psi_1 \varepsilon\, b_{22} \frac{\partial V}{\partial y} - \psi_1 \varepsilon\, b_{33} \frac{V_i}{y}} \qquad (4.5)$$

where

$$i = \begin{cases} 0 & \text{For the plane jet} \\ 1.0 & \text{For the round jet} \end{cases}$$

$$b_{ij} = \frac{\overline{u_i u_j}}{\overline{q^2}} - \frac{1}{3} \delta_{ij} \qquad (4.6)$$

$$C_o = \frac{2}{3C_1(2-b)} \qquad (4.7)$$

$$C_2 = \frac{2(C_1-2)}{20+C_1(8-9b)} \qquad (4.8)$$

The above equations have been arranged so that they can be easily applied to either the plane or the axisymmetric jet. Also the diffusion terms are arranged so that the terms that are of the gradient type will be treated implicitly in the numerical procedure, while the rest of the terms including the production and rapid terms will be treated as source terms. The underlined terms in the production and pressure strain part are second order terms based on the order of magnitude analysis of Appendix D. However, these terms might be of importance, in particular near the axis of symmetry where the leading terms in the production and pressure group vanish; hence these terms and also the second order terms in the mean momentum equations are retained.

4.3 Boundary Conditions

a) Outer Boundary

At the outer boundary all turbulence quantities and their derivatives (realizability condition) should vanish. The axial mean velocity also vanishes at the edge of the flow; the lateral component of the mean velocity will be determined from continuity equation at $y = y\ (\infty)$.

b) Inner Boundary

The values of $\overline{u_i u_j}$ and ε at the centerline of the jet are not known. The only source of information we have is that the flow considered is symmetric and hence we require that:

$$\begin{aligned} U(0) &= U_m \\ V(0) &= 0 \\ \overline{uv}(0) &= 0 \\ \frac{\partial \overline{u^2}}{\partial y}(0) &= 0 \\ \frac{\partial \overline{v^2}}{\partial y}(0) &= 0 \\ \frac{\partial \overline{w^2}}{\partial y}(0) &= 0 \\ \frac{\partial \varepsilon}{\partial y}(0) &= 0 \end{aligned} \qquad (4.9)$$

and also for the axisymmetric case we have

$$\overline{v^2}(0) = \overline{w^2}(0) \qquad (4.10)$$

Equations (4.9) could be used directly as boundary conditions in the numerical scheme, however, to maintain the control difference accuracy the continuity and momentum equation will be evaluated at the centerline incorporating the above conditions. The resulting set of equations will be solved simultaneously for $\overline{u^2}(o)$, $\overline{v^2}(o)$, $\overline{w^2}(o)$ and $\varepsilon(o)$ along with the difference equations evaluated at the other grid points. This will be discussed later when the equations of motion are transformed to similarity variables.

4.4 Similarity Solution

The similarity analysis has been discussed in Section (3.5). Now let us define the normalized kinetic energy of the turbulence and the dissipation by:

$$\varepsilon = U_m^3\, g_5(\eta)/\ell \qquad (4.11)$$

$$\overline{q^2} = U_m^2\, g_6(\eta) \qquad (4.12)$$

where $\eta = y/\ell\ (x)$ and $\xi = \sqrt{a_1}\eta$ which is used in the similarity equations.

If we substitute equations (3.18) - (3.20), (4.11) and (4.12) into the equations for mean and turbulent quantities we obtain the following set:

Continuity:

$$\frac{i+1}{2} f + \xi f' - \frac{1}{\xi^i}(\xi^i h)' = 0 \qquad (4.13)$$

Momentum:

$$\frac{i+1}{2} f^2 + \xi f f' - h f' - \frac{1}{\xi^i}(\xi^i g_4)' + (i+1)(g_1 - g_2) + \xi(g_1' - g_2') = 0 \qquad (4.14)$$

Reynolds Stresses:

$$(k_{mn} g_n')' + [\frac{i}{\xi^i}\ell_{mn} + (\xi f - h)\,\delta_{mn}]\, g_n' + [(i+1)\, f - C_1 \frac{g_5}{g_6}]\,\delta_{mn}\, g_n$$

$$+ \lambda_m\, g_5 = d_m \qquad (4.15)$$

where*

$$\delta_{mn} = \begin{cases} 0 & m \neq n \\ 1.0 & m = n \end{cases}, \quad \begin{matrix} n = 1,2,3,4 \\ m = 1,2,3,4 \end{matrix}$$

and where

$$k_{mn} = \left(\frac{C_o g_6}{g_5}\right) \begin{vmatrix} (C_2+1)g_2 & 3C_2g_2 & C_2g_2 & 2(C_2+1)g_4 \\ (C_2-\frac{2}{5})g_2 & 3(C_2+\frac{3}{5})g_2 & (C_2-\frac{2}{5})g_2 & 2(C_2+1)g_4 \\ C_2g_2 & 3C_2g_2 & (C_2+1)g_2 & 2C_2g_4 \\ -\frac{3}{5}g_4 & \frac{4}{5}g_4 & -\frac{1}{5}g_4 & \frac{8}{5}g_2 \end{vmatrix} \tag{4.16}$$

$$\ell_{mn} = \left(\frac{C_o g_6}{g_5}\right) \begin{vmatrix} C_2g_2 & 3(C_2+1)g_2 & (C_2-2)g_2 & 2C_2g_4 \\ C_2g_2 & 3(C_2+1)g_2 & (2-2)g_2 & 2C_2g_4 \\ (C_2-\frac{2}{5})g_2 & 3(C_2-\frac{2}{5})g_2 & (C_2+\frac{13}{5})g_2 & 2(C_2-\frac{2}{5})g_4 \\ 0 & g_4 & -g_4 & 2g_2 \end{vmatrix} \tag{4.17}$$

$$\lambda_m = \begin{vmatrix} \frac{1}{3}(C_1-2) \\ \frac{1}{3}(C_1-2) \\ \frac{1}{3}(C_1-2) \\ 0 \end{vmatrix} \tag{4.18}$$

*Repeated subscripts imply summation.

$$
\begin{aligned}
d_m = & - \left| \begin{matrix} - \frac{2}{3}\,(4C-1)g_4 \\ + \frac{4}{3}\,(1+5C)g_4 \\ 4Cg_4 \\ - \frac{7+50C}{15} g_1 + \frac{20C-2}{15} g_2 + \frac{2+20C}{10}\, g_3 \end{matrix} \right| f' \\
& + \left| \begin{matrix} -2g_1 + 4(\frac{1}{15} - cb_{11})g_6 \\ 4[- \frac{1}{30} + c(b_{11}+b_{22})]g_6 \\ 4[- \frac{1}{30} + c(b_{11}+b_{33})]g_6 \\ - (1+2C)g_4 \end{matrix} \right| \left(\frac{i+1}{2}\, f + \xi f'\right) \\
& - \left| \begin{matrix} 4[- \frac{1}{30} + c(b_{11}+ b_{33})]g_6 \\ 4[\frac{1}{15} - cb_{22}]g_6 - 2g_2 \\ 4[- \frac{1}{30} + c(b_{22}+b_{33})]g_6 \\ - (1+2c)g_4 \end{matrix} \right| h' - \left| \begin{matrix} 4[- \frac{1}{30} + c(b_{11}+b_{33})]g_6 \\ 4[- \frac{1}{30} + c(b_{22}+b_{33})]g_6 \\ -2g_3 + 4(\frac{1}{15} - cb_{33})g_6 \\ - 4cg_4 \end{matrix} \right| \frac{h}{\xi}\, i \\
& - \frac{i}{\xi} \frac{d}{d\xi} \left| \begin{matrix} 2\frac{C_o C_2}{g_5}\, g_6(g_2-g_3)g_3 \\ 2\frac{C_o g_6}{g_5}\,(C_2 - \frac{2}{5})(g_2-g_3)g_3 \\ 2\frac{C_o g_6}{g_5}\,(C_2+1)(g_2-g_3)g_3 \\ - \frac{2}{3}\,\frac{C_o g_6}{g_5}\, g_4 g_3 \end{matrix} \right| - \left| \begin{matrix} 0 \\ - \frac{16}{5}\,(g_2-g_3)g_3 \\ \frac{16}{5}\,(g_2-g_3)g_3 \\ - \frac{8}{5}\, g_4 g_3 \end{matrix} \right| \left(\frac{C_o g_6}{g_5}\right) \frac{i}{\xi^2}
\end{aligned}
\qquad (4.19)
$$

Dissipation "Rate" Equation:

$$(k_{55}g_5')' + (\frac{i}{\xi}\ell_{55} + \xi f - h)g_5' + (\frac{3i+5}{2} f - \psi_0 \frac{g_5}{g_6})g_5 = d_5 \quad (4.20)$$

where

$$k_{55} = \ell_{55} = \frac{9}{5(4C_1+9b+10)} (\frac{g_6}{g_5}) [g_2 + 2\frac{g_2^2+g_4^2}{g_6}] \quad (4.21)$$

$$d_5 = \psi_1 g_5[b_{12}f' - \frac{i+1}{2}(f+\xi f') + b_{22}h' + b_{33}\frac{h}{\xi} i \quad (4.22)$$

Centerline Values of the Reynolds Stress

By transforming the boundary condition (4.9) to similarity variables we obtain:

$$f(0) = 1 \quad (4.23a)$$

$$f'(0) = 0 \quad (4.23b)$$

$$h(0) = 0 \quad (4.23c)$$

$$g_1'(0) = 0 \quad (4.24a)$$

$$g_2'(0) = 0 \quad (4.24b)$$

$$g_3'(0) = 0 \quad (4.24c)$$

$$g_4(0) = 0 \quad (4.25a)$$

$$g_5'(0) = 0 \quad (4.25b)$$

When the continuity and momentum equations (4.13) and (4.14) are evaluated in the limit $\xi \to 0$, we get

$$h'(0) = \frac{1}{2} \quad (4.26a)$$

$$g_4'(0) = \frac{1}{2} + g_1(0) - g_2(0) \quad (4.26b)$$

Now let us evaluate the Reynolds stress equation at centerline. By substituting the above condition in equation (4.15) we get:

$$k'_{mn}\, g'_n + [k_{mn} + \xi^{1-i}\, \ell_{mn}] g''_n + [(i+1) - C_1 \frac{g_5}{g_6}]\, \delta_{mn}\, g_n + \lambda_m\, g_5 = d_m \tag{4.27}$$

For the shear stress component g_4 ($m=4$ in the above equation) all that is left is:

$$g''_4(0) = 0 \tag{4.28}$$

On the other hand, for the normal stresses g_1, g_2 and g_3 ($m=1,2,3$, and $n=1,2,3,$), equation (4.27) becomes:

$$[k_{mn} + \xi^{1-i}\, \ell_{mn}]\, g''_n + [(i+1) - C_1 \frac{g_5}{g_6}]\, \delta_{mn}\, g_n + \lambda_m\, g_5 = dm \tag{4.29}$$

The coefficients k_{mn}, ℓ_{mn} and λ_m and the source terms dm will be evaluated from (4.16-4.19) by letting $\xi \to o$. The dissipation equation when evaluated at $\xi = o$ takes the form:

$$[k_{55} + \xi^{1-i}\, \ell_{55}]\, g''_5 + [\frac{3i+5}{2} - \psi_o \frac{g_5}{g_6}] g_5 = d_5 \tag{4.30}$$

where k_{55}, ℓ_{55} and d_5 are obtained from equations (4.21) and (4.22). Hence equations (3.29) and (3.30) provide four equations which can be solved simultaneously to obtain the centerline values of the normal stresses g_1, g_2, and g_3 together with the dissipation rate of the turbulence kinetic energy g_5.

4.5 Model's Parameters

The kinetic energy decay rate function ψ_o and the return to isotropy parameters are given by equation (2.27) and (2.46) which can be rewritten as:

$$\psi_o = \frac{14}{5} + .98[1-\ln(1-55II)] \exp(-2.83 R_\ell^{-.5}) \qquad (4.31)$$

$$C_1 = 2.0 + (\frac{1}{9} + 3\ III + II)[72 R_\ell^{-.5}$$

$$+ 80.\ln(1+ 62.4\ II + 2.3\ III)] \exp(-7.77 R_\ell^{-.5}) \qquad (4.32)$$

The turbulence Reynolds number R_ℓ is defined by equation (2.17). It can be written in terms of the similarity variables as follows:

$$R_\ell = \frac{U_m \ell}{\nu} (\frac{g_6^2}{9g_5}) \qquad (4.32)$$

If we substitute for the centerline mean velocity U_m and the turbulence length scale ℓ their respective similarity definitions (3.26 and 3.42), equation (4.32) becomes:

$$R_\ell = \frac{U_o d_o}{\nu} \cdot \frac{a_1 C}{9} \cdot \frac{g_6^2}{g_5} \cdot (\frac{x}{d_o})^{\frac{1-i}{2}} \qquad (4.33)$$

or

$$R_\ell = \frac{a_1 C}{9} \cdot \frac{g_6^2}{g_5} \cdot Re \cdot (\frac{x}{d_o})^{\frac{1-i}{2}} \qquad (4.34)$$

where Re is the jet exit Reynolds number.

As we can see from the proceeding equation, the turbulence Reynolds number will depend on the jet exit Reynolds number for both the plane and round jets. Furthermore, R_ℓ will be a function of x in the two-dimensional flow case. Hence ψ_o and C_1 will be functions of the downstream position. This of course, contradicts the similarity formulation which assumes that all variables in question will be functions of the similarity variable η only. It turns out, however, in the jet calculation, that ψ_0 and C_1 dependence on the R_ℓ is rather weak. It has been observed in the present calculation that changing the turbulence Reynolds number by 50% only changes the final turbulence profiles by 1-2% with hardly any change in the mean velocity profiles.

For the present calculations we let

$$R_\ell = RET\left(\frac{g_6^2}{g_5}\right) \tag{4.35}$$

where

$$RET = \frac{Ca_1}{9}\left(\frac{x}{d_o}\right)^{\frac{1-i}{2}} \frac{U_o d_o}{\nu} \tag{4.36}$$

and RET is kept constant for either case plane or round. For the round jet, x disappears from the above equation, while in the plane jet we simply choose an average value for $\frac{x}{d_o}$ from the experimental data (see Table 4.1).

The remaining parameters of this model are ψ_1 which appears in the production term of ε-equation, c in the rapid terms and finally b, the relaxation parameter in the diffusion terms. These, however, will be taken constant in the current calculation, and assigned the values that are given in Table 4.1.

Flow	x/d_o	C	a_1	$R_\ell = \frac{U_o d_o}{\nu}$	ψ_1	c	b
Round Jet	-	6.0	.09	87000	2.0	-.15	0
Plane Jet	100	2.4	.11	24000	2.0	-.15	0

Table 4.1 Reynolds Stress Model Constants

The jet exit Reynolds number was based on Rodi (1975) experiment for the round jet, and on Heskestad (1965) for the plane jet. The constant in the centerline decay law C and the jet growth rate a_1 are assigned the values shown in Table (4.1) which are average values of the observed data.

4.6 Numerical Solution

The system of equations (4.13), (4.14), (4.15) and (4.20) constitutes a closed set of ordinary differential equations which are sufficient to solve the seven unknowns, U, V, $\overline{u^2}$, $\overline{v^2}$, $\overline{w^2}$, $\overline{uv}$ and ε. However, solving these equations is not a trivial matter, and the outcome of the solution will depend on how the various terms in stress and dissipation equations are treated in the computation. For example, in the application of the k-ε model, the production terms have been linearized, and treated implicitly in the calculation; this worked fairly well. On the other hand, the production and pressure strain terms in the stress model which include the mean velocity gradients, involve a lot of terms as compared to the k-ε model. Hence it seems to be more reasonable for the computation to treat these terms explicitly and add them to the source terms. All the other terms that involve the gradients of the Reynolds stress components and the dissipation terms will remain implicit in the calculations.

The system of equations cited above are quasi-linearized in a way similar to that used in the k-ε model solution. They can be written as:

$$\frac{(i+1)}{2}\xi^{i}\,f + \xi^{i+1}\,f' - (\xi^{i}h)' = 0 \tag{4.37a}$$

$$[(i+1)\xi^{i}f_o + \xi^{i+1}f_o^{1}]f + (\xi^{i+1}f_o - h_o\xi^{i})f' - \xi^{i}f_o'h$$

$$+ (i+1)\xi^{i}(g_1-g_2) + \xi^{i+1}(g_1'-g_2') - (\xi^{i}g_4)'$$

$$= \frac{i+1}{2}\xi^{i}f_o^{2} + \xi^{i+1}f_of_o^{1} - \xi^{i}h_of_o' \tag{4.37b}$$

$$(k_{mn}g_n')' + [i\xi^{-1}\ell_{mn} + (\xi f-h)\delta_{mn}]_o\,g_n'$$

$$+ [(i+1)f - C_1\frac{g_5}{g_6}]_o\,\delta_{mn}\,g_n + (\lambda_m)_o\,g_5 = (d_m)_o \tag{4.37c}$$

$$(k_{55}g_5')' + [i\xi^{-i}\ell_{55} + \xi f-h]_o\,g_5' + [\frac{3_i+5}{2}f - \psi_o\frac{g_5}{g_6}]_o\,g_5 = (d_5)_o \tag{4.37d}$$

where the "o" subscript indicates that the quantity is fixed for that cycle of the iteration process and evaluated from the previous iteration. In the early stages of the computation the stress equation (4.37c) and the dissipation equation (4.37d) were solved simultaneously for the stress components and the dissipation rate; then with the shear and normal stresses known, the momentum (4.37b) and continuity (4.37a) equations were solved for the mean flow velocity components U and V. It was, however, difficult to get a smooth mean velocity profile near the centerline. This is because that when the gradient of the shear stress(which is the most dominant term in the momentum equation) increases or decreases slightly, the resulting change in the mean velocity profile is noticeable. It was decided to solve the set of equations (4.37) simultaneously using the difference scheme of Appendix F. This method solves for the unknowns at three nodes point simultaneously, as previously mentioned in the k-ε model application.

To start the computation, the eddy viscosity solution in Appendix C was used as an initial guess with $\overline{u^2} = \overline{v^2} = \overline{w^2} = \frac{1}{3}\overline{q^2}$. Those profiles, however, were taken for $\xi \geq \xi_{ss}$ (see Appendix C). For $\xi < \xi_{ss}$ the profiles were approximated so that they take values reasonably close to experimental data at the centerline of the jet. The model parameters C_1 and ψ_o were initially assigned the values (3.25)and (3.8) respectively, and then they were updated using equation (4.31) and (4.32) after each iteration.

4.7 Results

The calculated mean and turbulent profiles are plotted versus the similarity variable ($\xi=\frac{r}{x}$) and are shown in the figures 4.1-4.15. The results are compared with the most recent data of Abbiss et al. (1975), Rodi (1975) and Wygnanski and Rodi (1969) for the round case. For the plane jet comparisons the data of Bradbury (1965), Heskestad (1965) and Gutmark and Wygnanski (1976) are used. Table 4.2 summarizes the flow constants of turbulent free jets for both plane and axisymmetric flows.

a) Mean Velocities

Figure 4.1 and Figure 4.6 show the calculated distribution of the normalized axial components of the mean velocities for both plane and round jets while figure 4.11 displays the distribution of the normalized lateral component of the mean velocities across self-preserving jets. Not much need to be said about the shape of the mean profiles because their behavior is well known and they agree fairly well with the data within the experimental accuracy of the measured values.

If we now take a closer look at the mean velocity profiles and test whether the momentum integral constraint (3.27) is satisfied or not then we need to bring the decay rates of the mean centerline velocity of the jet into the picture. (See equation B25). This is an important point which has been raised by Baker (1980) (see also George et al. 1981).

For example, Figure 3 of Wygnanski and Fiedler's experiments (1969) gives C = 5.0 while in Rodi's latest work (1975)Figure 8 gives C = 6.0. On the other hand their normalized mean velocity profiles are nearly identical including the spreading rate. (see Table 4.2). Obviously if one of the above cited experiments satisfies conservation of momentum, the other will not. This point will be discussed in some detail in Chapter 5.

Flow	Reference	$\frac{d\ell}{dx}$	C	$R = \frac{U_o d_o}{\nu}$	$(\frac{\overline{uv}}{U_m})_{max}$	$\frac{x}{d_o}$
Plane Jet	Bradbury (1965)*	not constant	2.4	$3x10^4$	.0242	14-70
	Heskestad (1965)	.11	-	$.47x10^4$ -$3.7x10^4$	.021	47-155
	Gutmark & Wygnanski (1976)	.102	2.306	$3x10^4$	.024	120
	Present Results k-ε model	.1106	2.46	-	.022	-
	Reynolds stress	.112	2.42	$.47x10^4$		100
Round Jet	Wygnanski & Fiedler (1969)	.086	5.0	10^5	.0165	20-98
	Rodi (1975)	.086-.09	6.0	$8.7x10^4$	.0186	62-75
	Abbiss et al.	.89-.1	5.5	$5.75x10^4$	.0221	20-30
	Present Results k-ε model	.087	6.4	-	.021	-
	Stress model	.095	5.8	$8.7x10^4$	.0198	-

Table 4.2 Flow Constants for Turbulent Jet Issuing in Still Air.

*For Bradbury's experiment U_E/U_m = .16, where U_E is the free stream velocity at the outer edge of the jet.

b) Stress Components

Figures 4.2, 4.3 and 4.4 show the variation of the normal components of the Reynolds stress across self-preserving axisymmetric jets. The calculated profiles are seen to be in fair agreement, within a few percent the experimental data of Wygnanski and Fiedler (1969) and Rodi (1975), hereafter referred to as reference I and reference II respectively. A little disagreement between the data of reference I and II is seen in lateral and azimuthal components of the Reynolds stress in particular near the jet axis. It seems also from figure 4.2-4.4 that either the calculated profiles are overestimated or the measured data are underestimating the result at the outer edge. The shear stress profile for the round jet is compared with the data of reference I and II and Abbiss et al. in figure 4.5. The Reynolds stress similarity solution gives higher value of the maximum shear stress as compared with I and lies between the values of reference II and the data of Abbiss et al. Otherwise the calculated shear stress for the round jet is well behaved over the entire region.

On the other hand the normal stresses $\overline{u^2}/U_m^2$, $\overline{v^2}/U_m^2$, $\overline{w^2}/U_m^2$ for the plane jet are in disagreement in the region for $\xi \leq .1$ as seen in the figures 4.7, 4.8 and 4.9. For example, in the central core of the jet the values observed by Bradbury (1965) of $\overline{u^2}/U_m^2$ is lower by as much as 50% than that of Gutmark and Wygnanski (1976) and about 25% lower than the data of Heskestad (1965). On the other hand, Bradbury observed a value of $\overline{v^2}/U_m^2$ which is 30% higher than that given by Gutmark and Wygnanski. Here we have to note that Bradbury's jet exhausted into a parallel stream which was moving at .16 of the jet exhaust velocity. So if we exclude Bradbury's result from comparison the discrepancies between Heskestad and Gutmark and Wygnanski will be sufficient to raise the question about the reliability of the data. The calculated profile as seen in the figures

4.7-4.9 agree well with the data for $\xi > .1$ and takes an average value of the above data for $\xi \leq .1$. However, we recall that the agreement between the data and calculated profiles for the turbulence kinetic energy $(k = \frac{1}{2} (\overline{u^2}+\overline{v^2}+\overline{w^2})/U_m^2$ was fair in the k-ε model result. The turbulent shear stress for the plane jet is shown in figure 4.10. The above data are in fair agreement while the similarity solution underestimate the shear stress near $\xi = .1$ but agrees well with the data near the centerline of the jet and for $\xi > .1$.

c) <u>Dissipation Rate "ε"</u>

For the plane jet the dissipation level of the kinetic energy of turbulence as predicted by the k-ε model and Reynolds stress similarity solution are in good agreement as seen in figure 4.14. We recall that energy and dissipation rate equations have been shown to be well balanced (Chapter 3). For the round jet the calculated dissipation rate using the k-ε model is higher at the centerline of the jet and lower for $\xi \geq \xi_{ss}$ than that given by the Reynolds stress solutions. This error in the dissipation rate for the k-ε model was a result of the model constants as mentioned earlier.

d) <u>Model Parameters</u>

The calculated values of the function ψ_0 for the decay rate of turbulence kinetic energy and the parameter C_1 in the return to isotropy term of the Reynolds stress equations are shown in figure 4.12 for both plane and axisymmetric self-similar jets. As it can be seen from figure 4.12, ψ_0 decreases slightly as we go toward the outer edge and it is nearly equal for both plane and round jet. In the same figure the variation of C_1 is shown, which, in fact, shows an interesting behavior. At the jet axis for both cases, plane and round jet, C_1 takes a value of about 3.5 which is close to the value suggested by Lumley (1978). Then C_1 increases to a maximum of

4.8 for the round jet and 6.4 for the plane jet at $\xi \cong .08$. The variation of the turbulent eddy viscosity ($\nu_t \sim \overline{q^2}^2/\varepsilon$)across the jet is shown in Figure 4.13. The shape of ν_t as obtained from the stress similarity solution is somewhat similar to that obtained in the k-ε model solution except here ν_t is seen to decrease at about $\xi \cong .1$ while in the k-ε model formulation the ratio $\overline{q^2}^2/\varepsilon$ decreases at about $\xi \cong .2$ as both $\overline{q^2}$ and ε vanish at the outer edge of the jet. The C_1 decreases only gradually as $\xi \to \infty$. The difference in magnitude of C_1 for the plane and axisymmetric jet is due to the variation of first and second invariant of the isotropic tensors (see figure 2.4).

4.8 Conclusion

The mean and turbulent profiles obtained using the Reynolds stress display satisfactory behavior in both flows, plane and round jets when they were compared with the data. The model constants ψ_1 and c has been taken as suggested by Reynolds (1976), namely $\psi_1 = 2$ and $c = -.15$. This choice of ψ_1 and c seems to work best for the turbulent jet. A similar behavior has been observed in the turbulent wake calculations (see Taulbee and Lumley 1980). If we increase ψ_1 slightly the kinetic energy of turbulence increases while the turbulent shear stress decreases. On the other hand c controls the pressure strain terms in the Reynolds stress equation. Changing c slightly is seen to affect the normal components of the Reynolds stress more than the turbulent shear stress. In previous theoretical work, Launder and Morse (1976) found that the spreading rate of the jet is 50% higher than the observed values. However Launder and Morse used only constants in their Reynolds stress model. They indicate that the difficulty in the computation arises from dissipation euqations. In the current calculation this behavior is not observed. The spreading rate which has been calculated using the Reynolds stress similarity solution is somewhat higher than the data but not by far as much as it is found in Launder and Morse's calculations.

The discrepancies in Launder and Morse's results might not be attributed to the source-sink term in the dissipation equation as indicated by the authors,but is likely due to their model for the transport terms.

The results of the present model which is well behaved for both plane and round jets indicate that the return to isotropy function C_1 which is used in the diffusion transport as well,varies considerably across the jet (see Figure 4.12),while the Launder and Morse model uses a pure constant for the diffusion transport which is based on computer optimization.

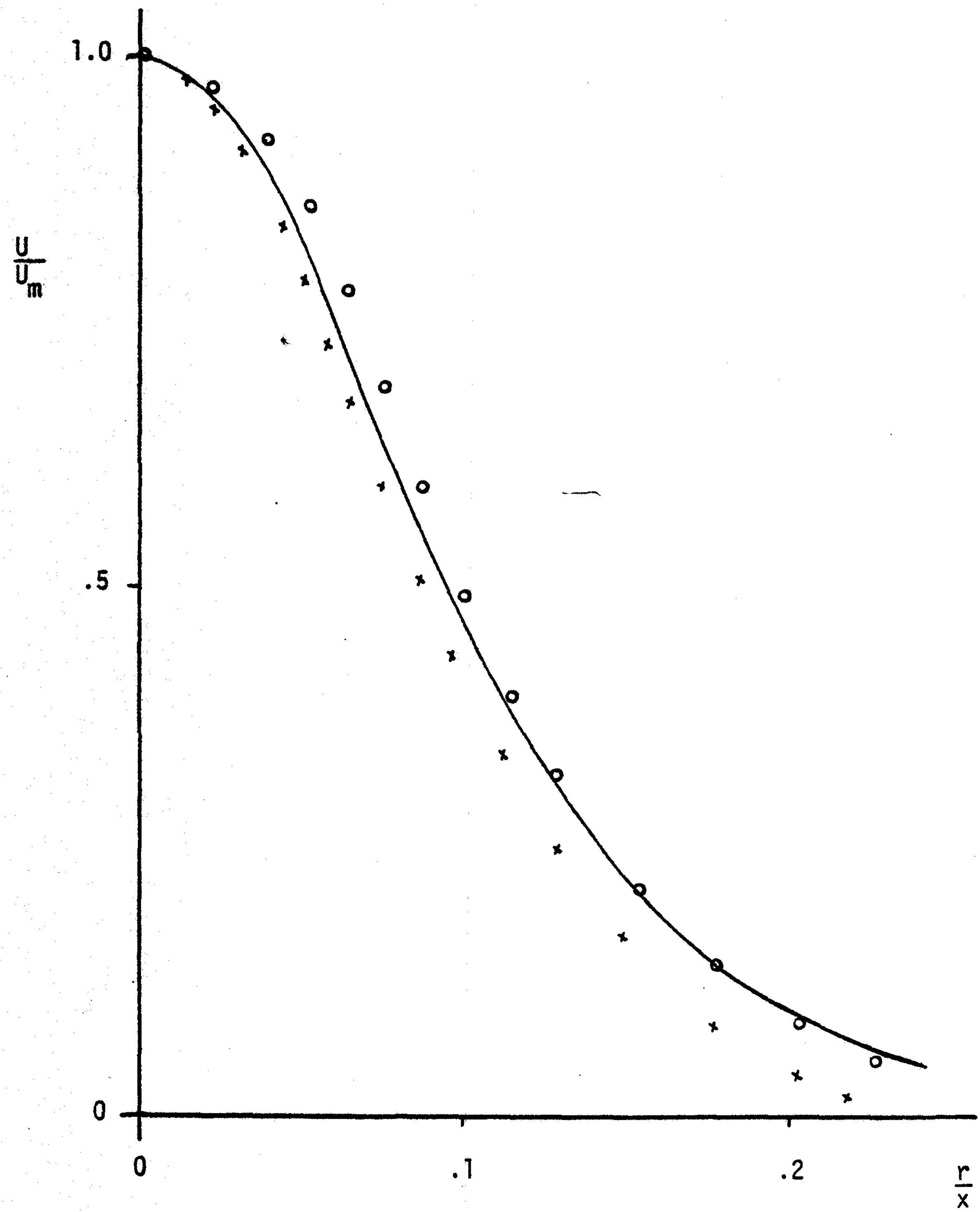

Figure 4.1 Mean Velocity Profile of Turbulent Round Jet

(o) Abbiss et al. (Pulsed wire)(1975)
(x) Wygnanski and Fiedler (1969)
(-) Reynolds Stress Similarity Solution

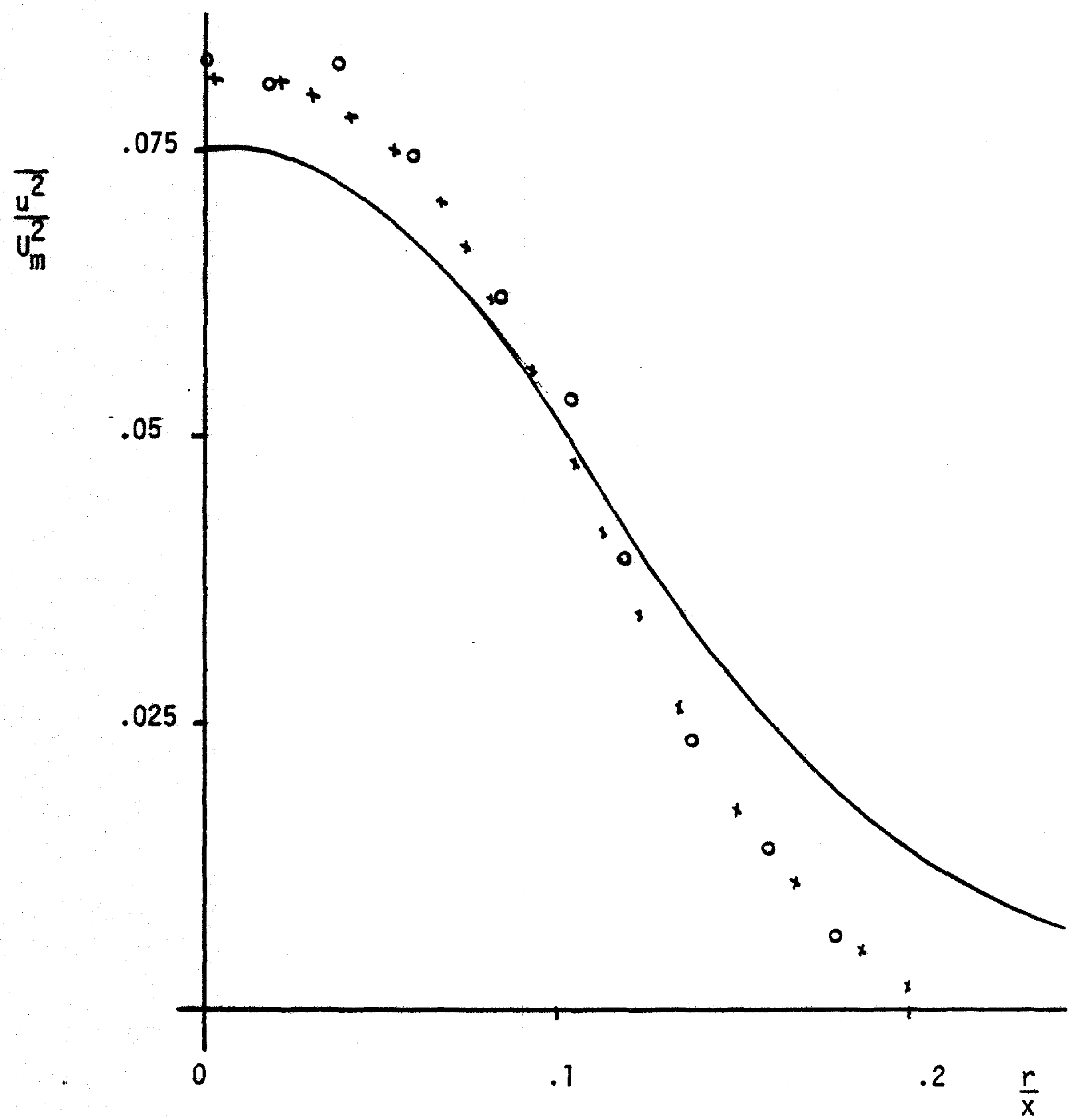

Figure 4.2 Axial Component of Reynolds Stress for the Round Jet

(o) Rodi (1975)
(x) Wygnanski and Fiedler (1969)
(-) Reynolds Stress Similarity Solution.

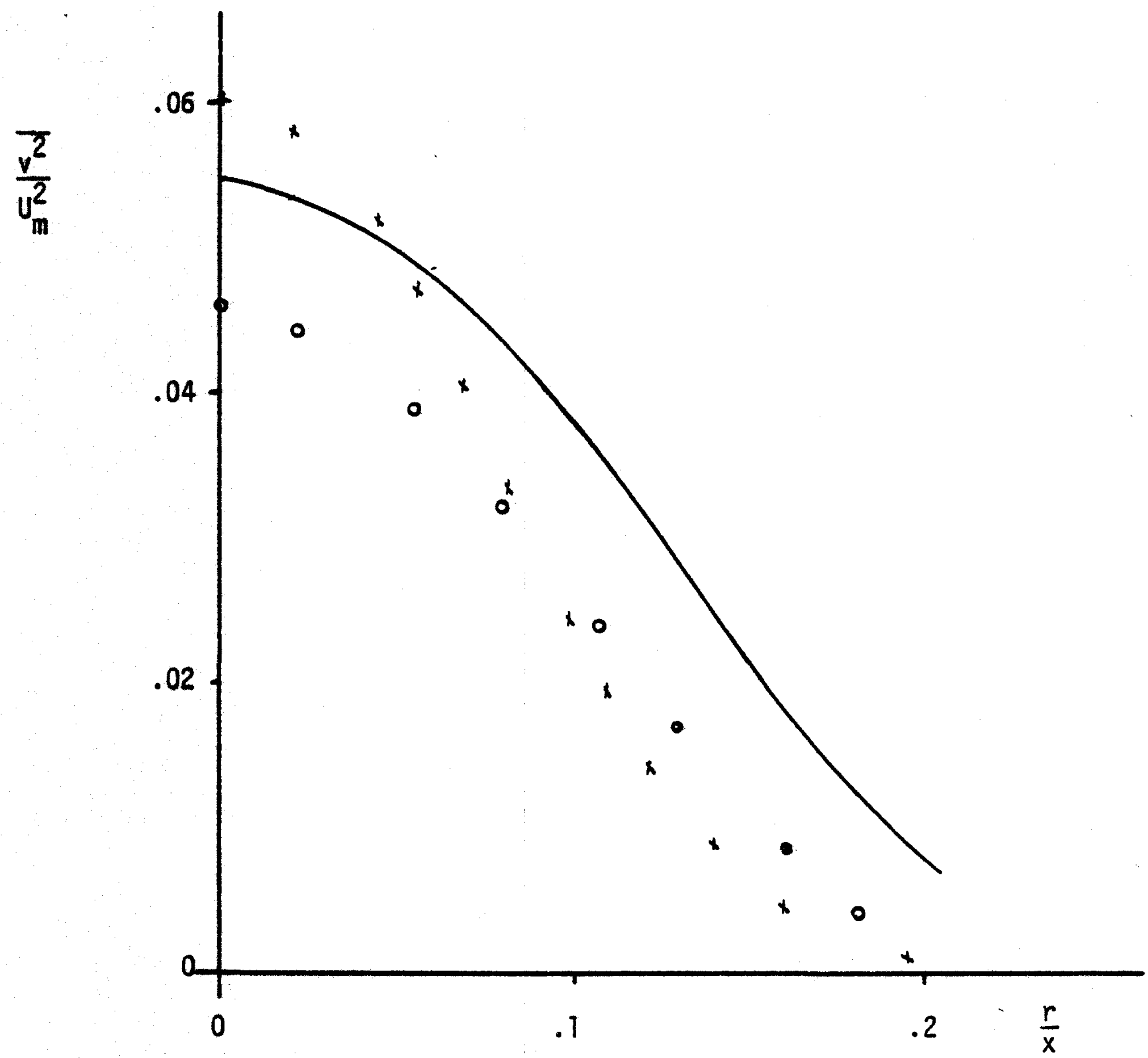

Figure 4.3 Radial Component of Reynolds Stress for the Round Jet. Notation as in Figure 4.2

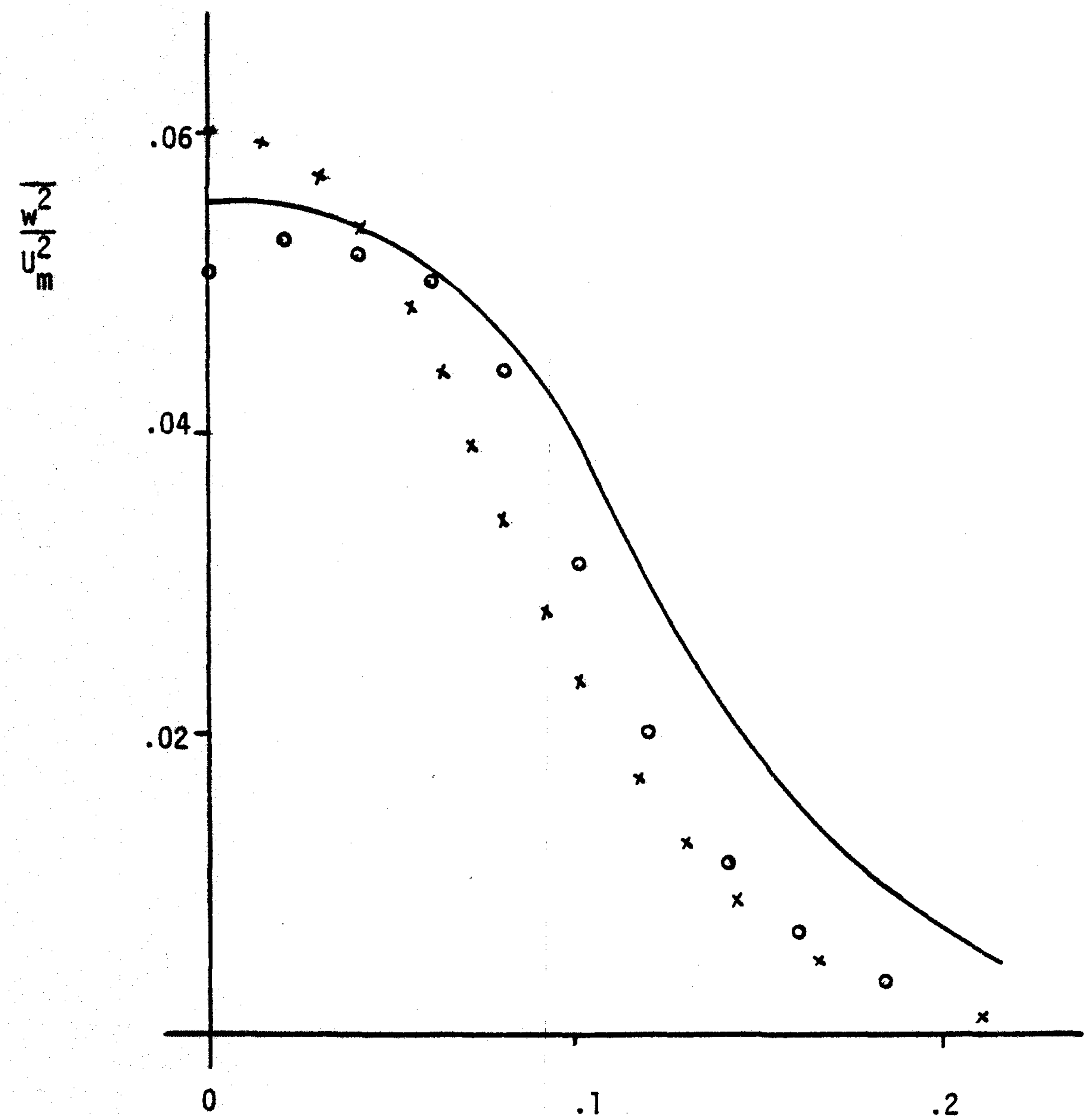

Figure 4.4 Azimuthal Component of Reynolds Stress for the Round Jet. Notations are the same as in Figure 4.2.

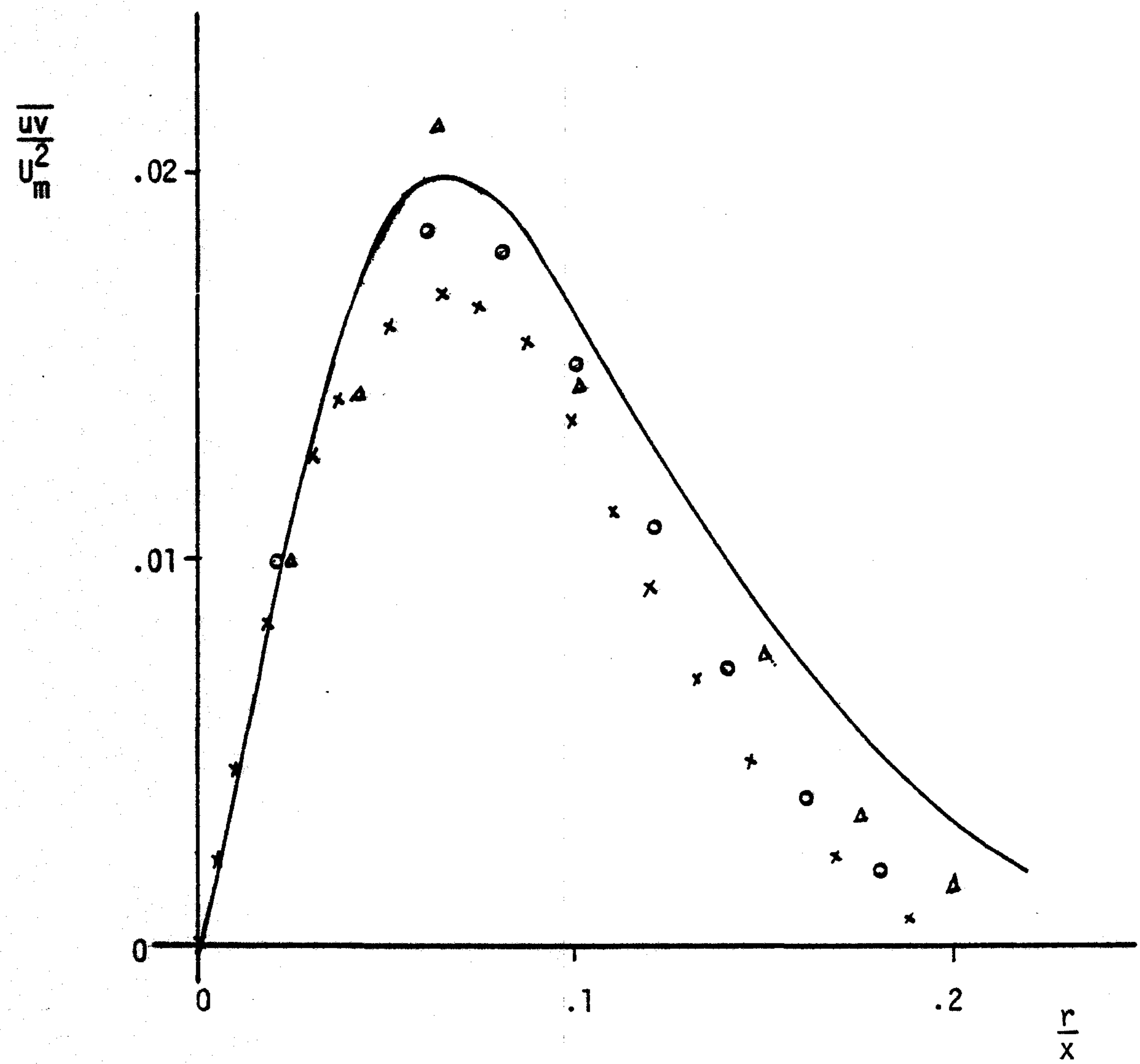

Figure 4.5 Shear Stress Profile for the Round Jet

(o) Rodi (1975)
(Δ) Abbiss et al. (Pulsed wire)(1975)
(x) Wygnanski and Fiedler (1969)
(——) Reynolds Stress Similarity Solution

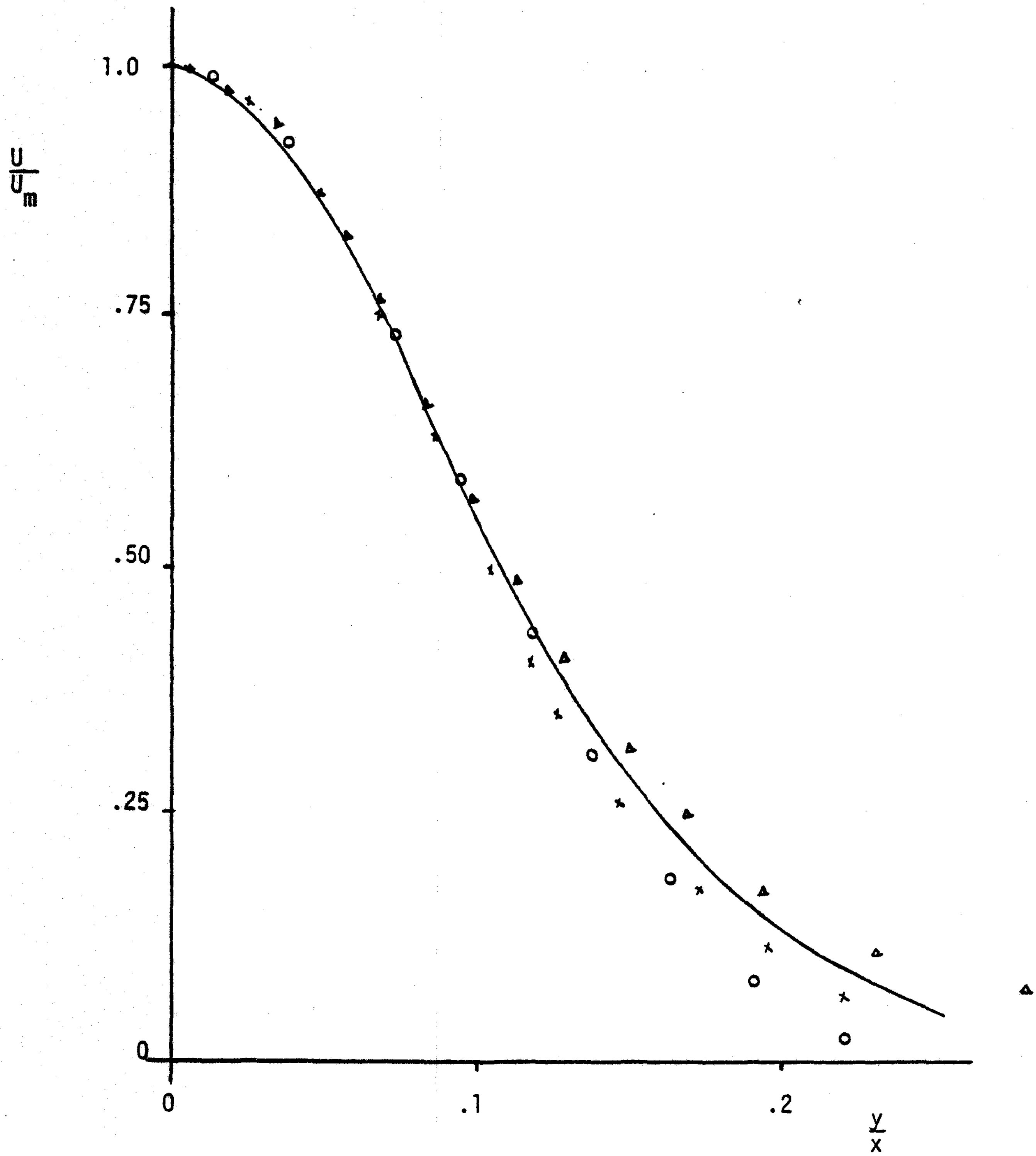

Figure 4.6 Mean Velocity Profile for Plane Jet

(o) Bradbury (1965)
(Δ) Heskestad(1965)
(x) Gutmark and Wygnanski (1976)
(-) Reynolds Stress Similarity Solution

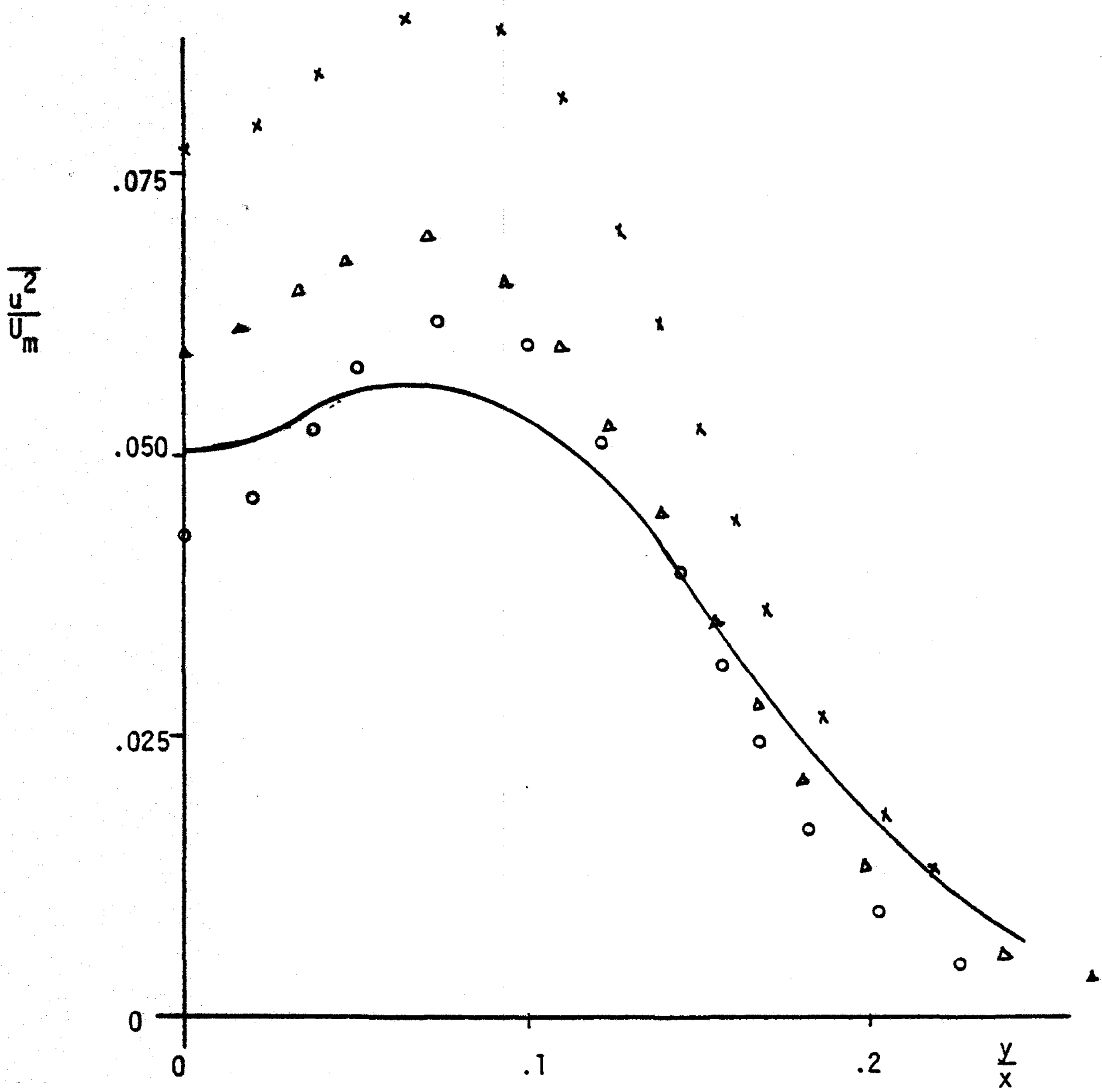

Figure 4.7 Axial Component of the Reynolds Stress for Plane Jet.

Notations are the same as in Figure 4.6.

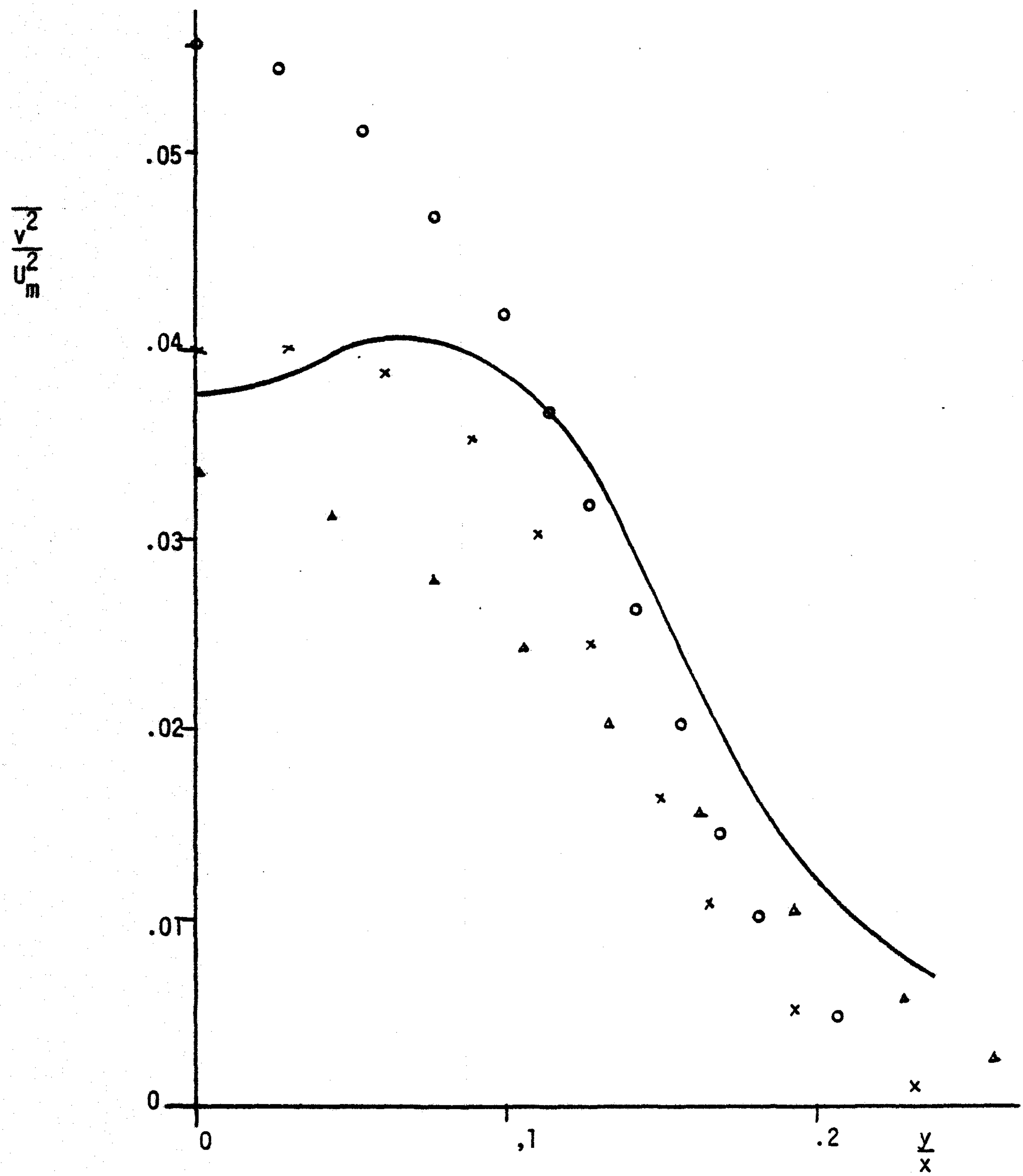

Figure 4.8 Vertical Component of the Reynolds Stress for Plane Jet.

Notations are the same as in Figure 4.6

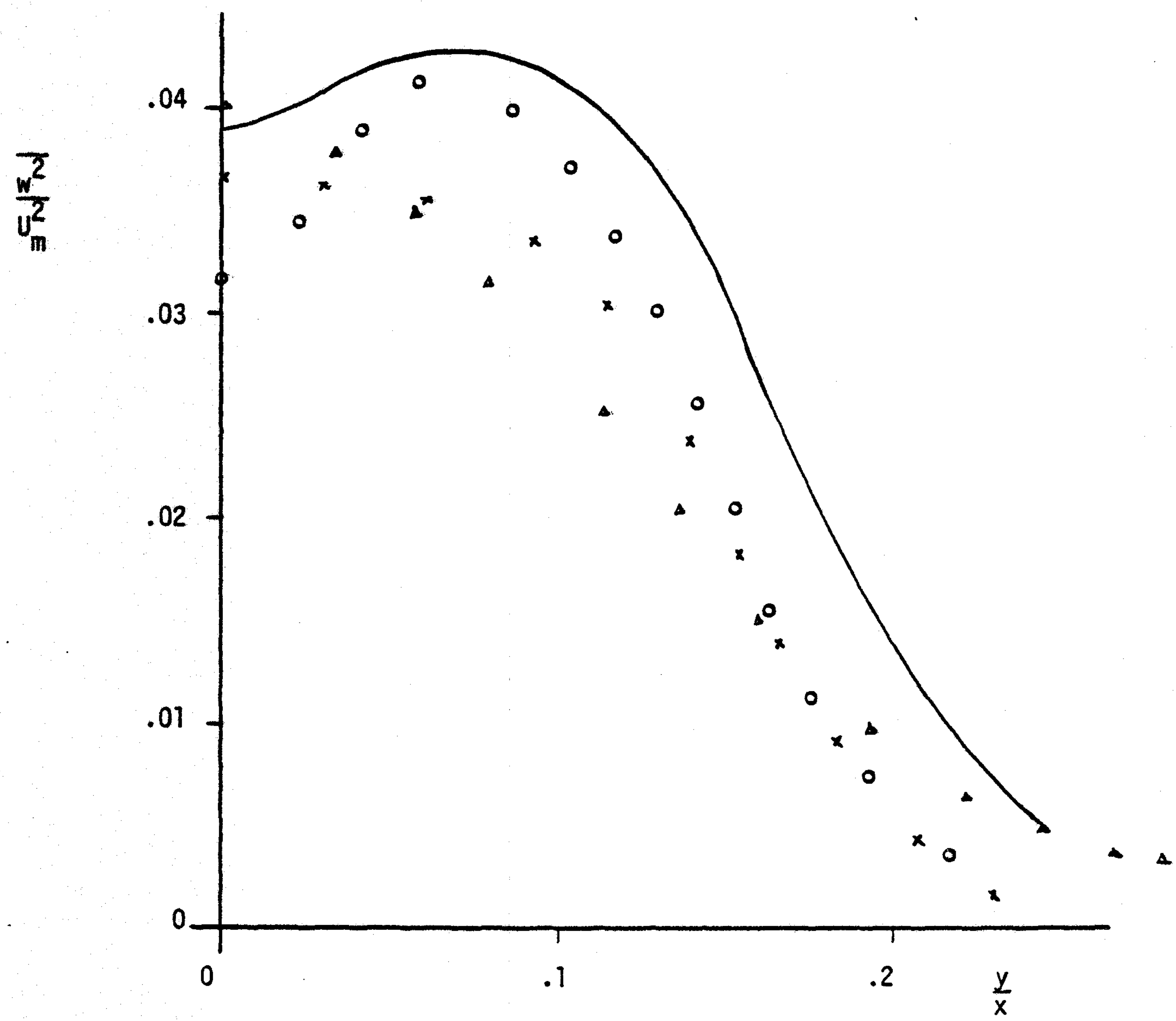

Figure 4.9 Horizontal Component of the Reynolds Stress for Plane Jet.

Notations are the same as in Figure 4.6

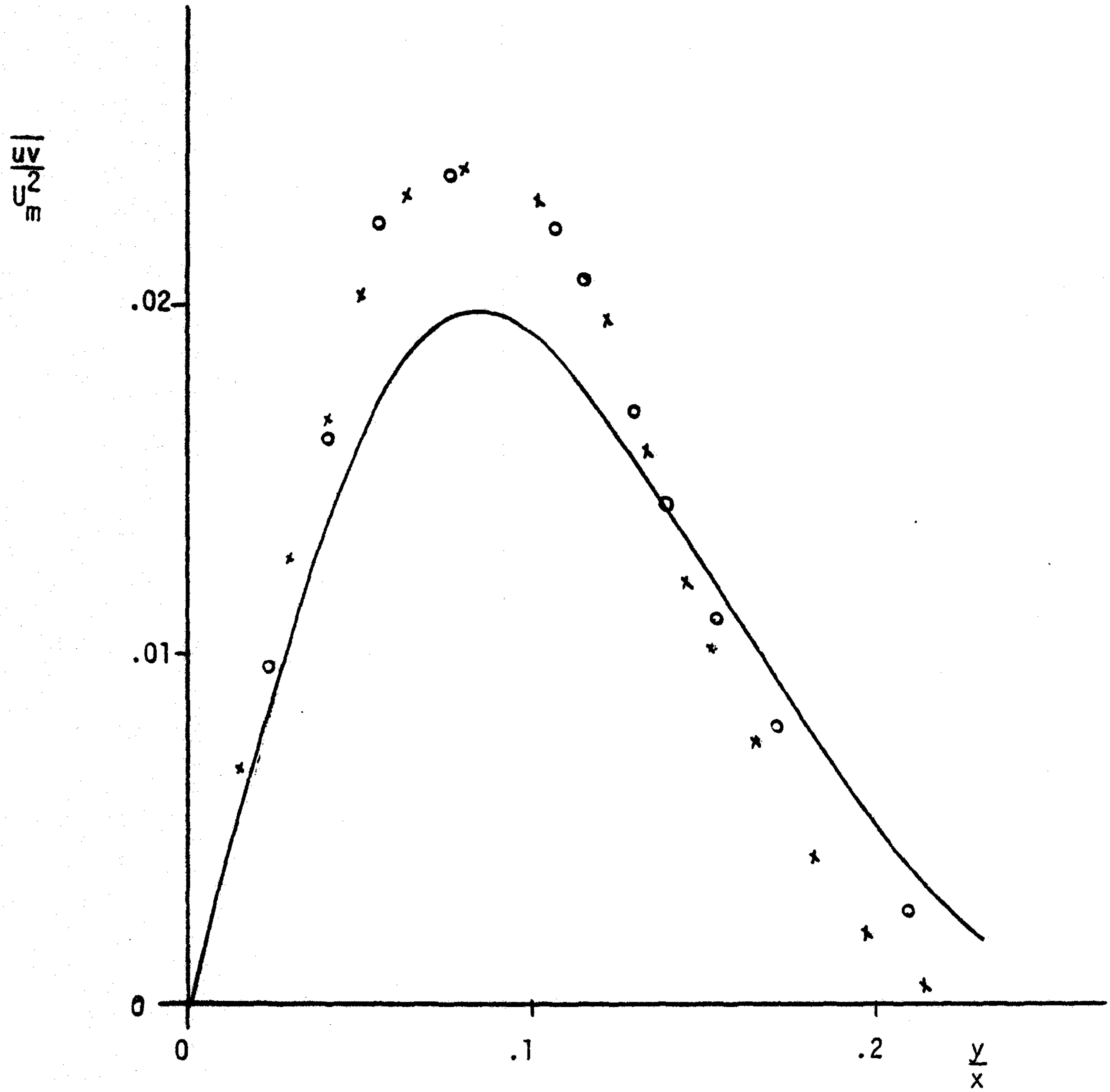

Figure 4.10 Shear Stress for Plane Jet

The notations are the same as in Figure 4.6

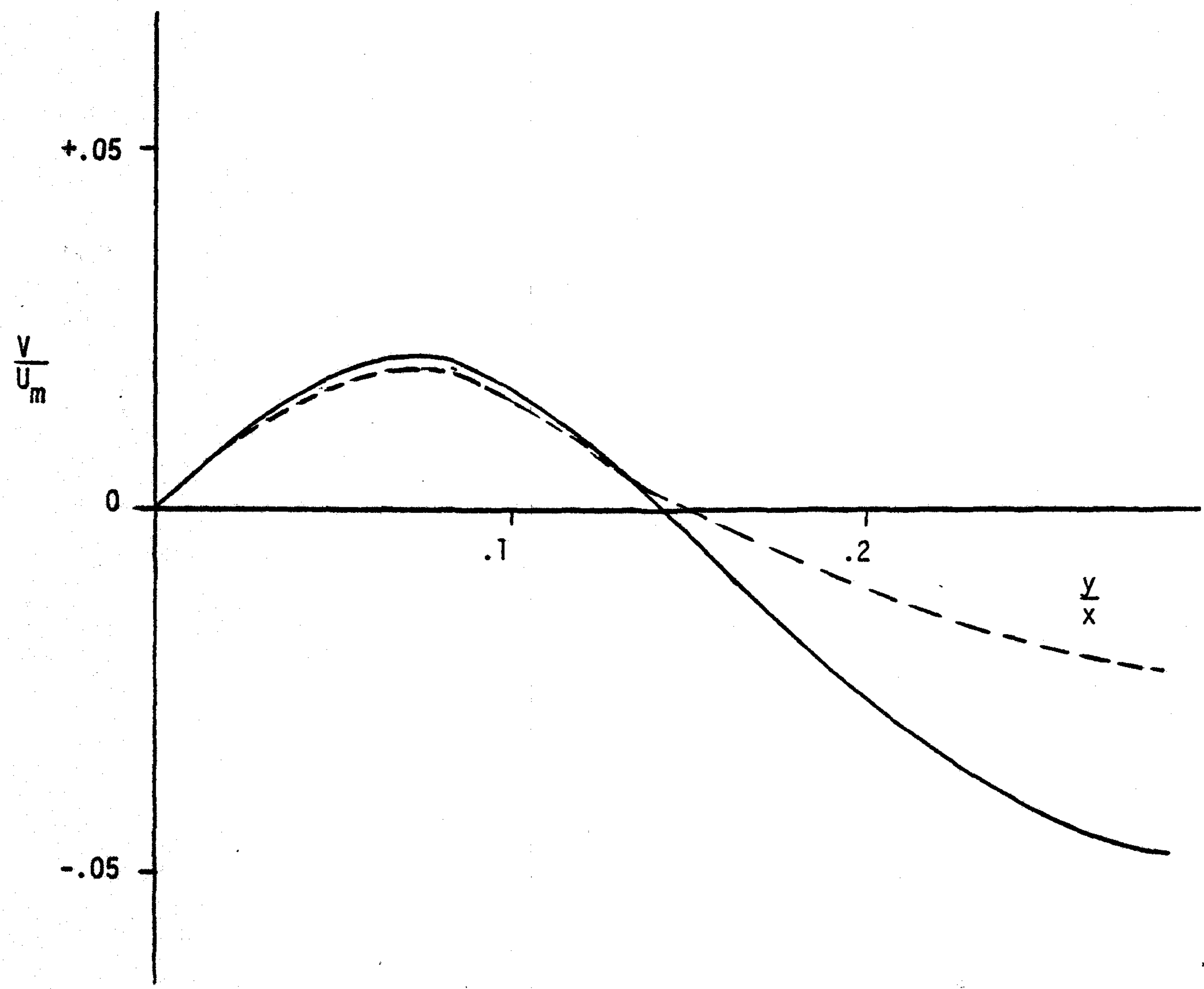

Figure 4.11 Radial Mean Velocity Profile of Turbulent Free Jets
(——) Plane Jet, (---) Round Jet

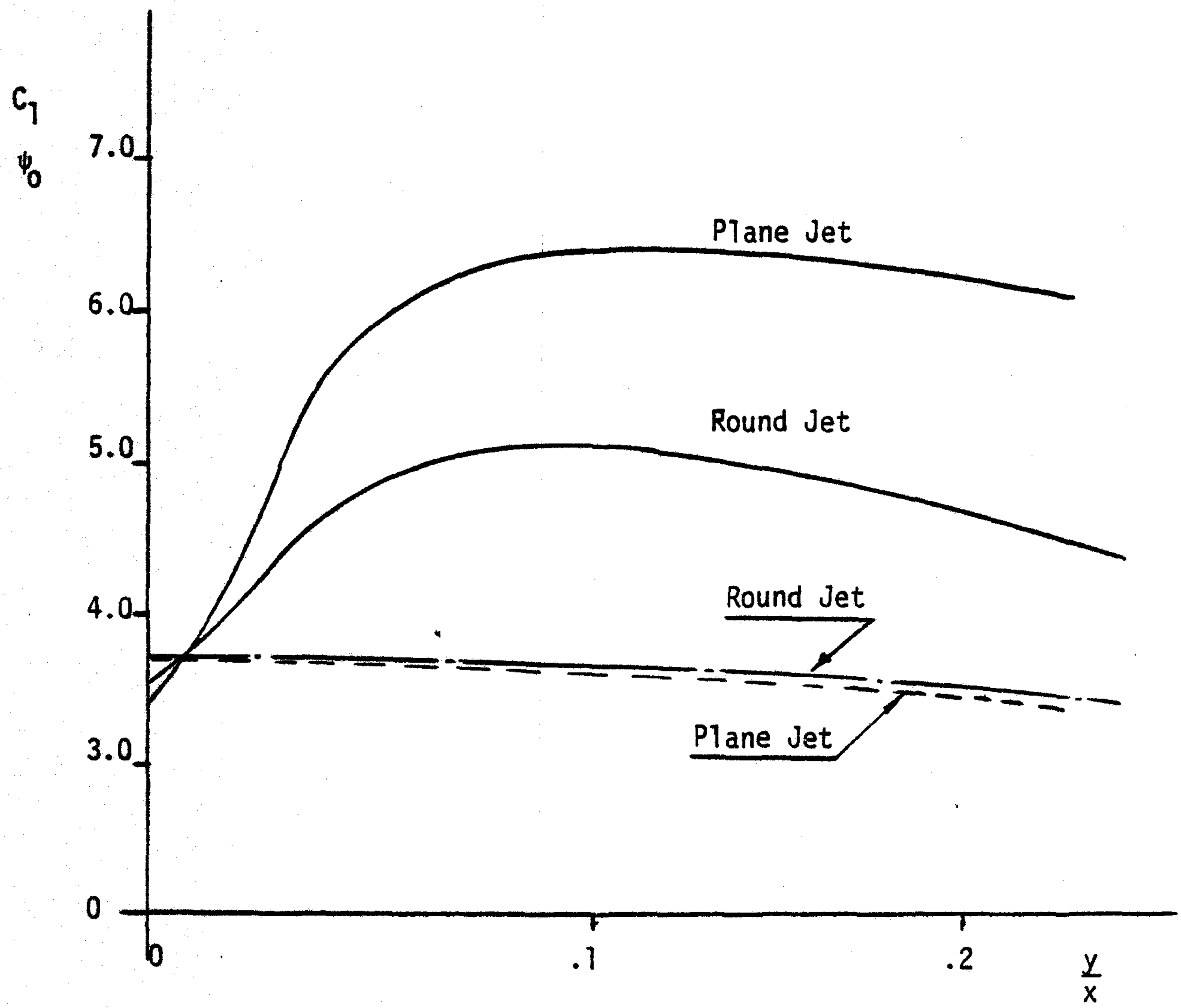

Figure 4.12 Variation of Reynold Stress Model Parameters Across Self-preserving Turbulent Jet

(——) The return to isotropy function (C_1)
(-·-), (----) Kinetic energy decay function ψ_0

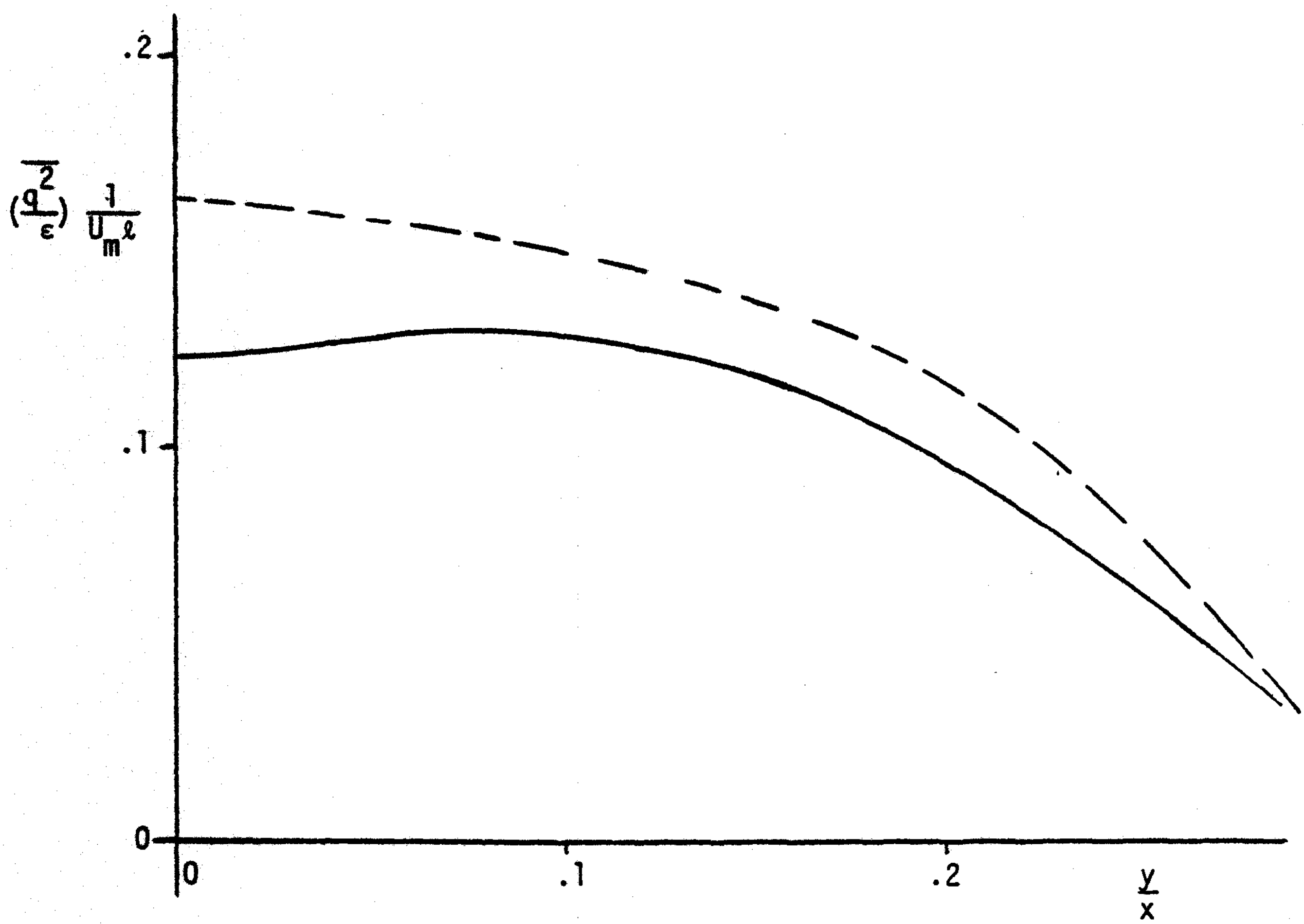

Figure 4.13 Calculated Variation of Turbulent Eddy Viscosity Across the Self-preserving Jet
(---) Round Jet, (——) Plane Jet

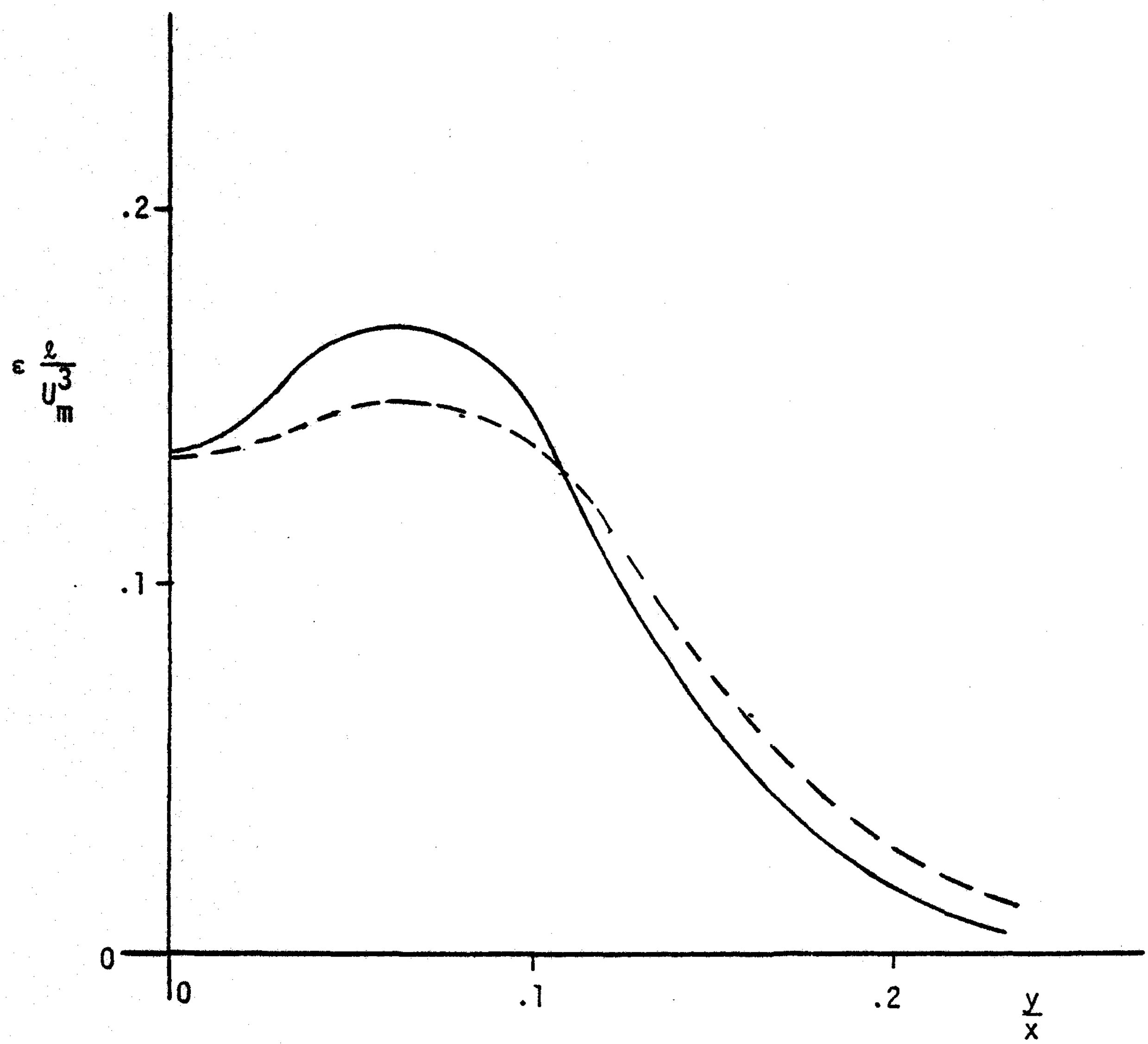

Figure 4.14 Calculated Dissipation Rate of Turbulence Kinetic Energy Across Plane Jet.

(——) (k-ε) model similarity solution
(---) Reynolds stress similarity solution

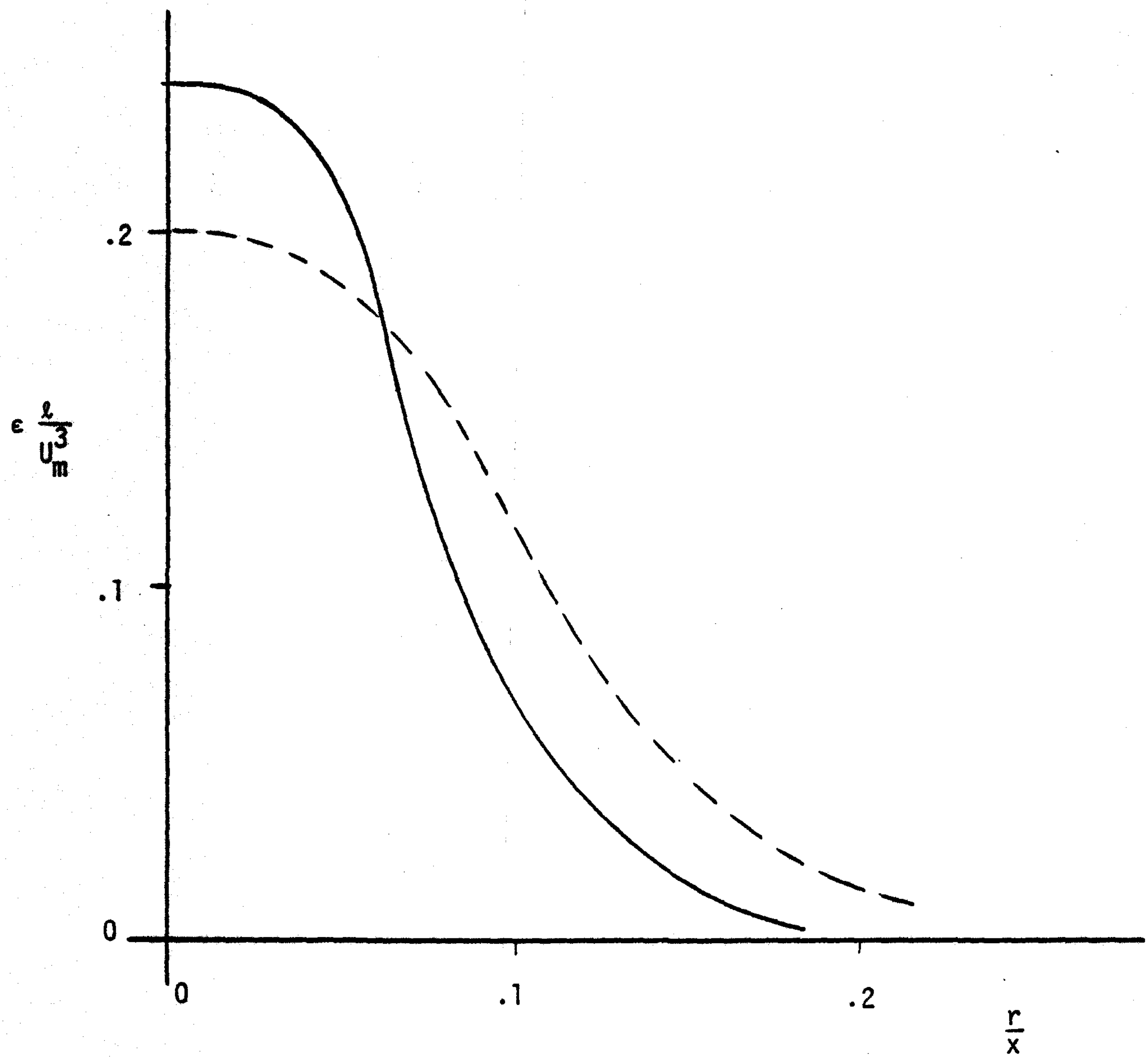

Figure 4.15 Calculated Dissipation Rate of Kinetic Energy of Turbulence for Axisymmetric Self-similar Turbulent Jet

(——) (k-ε) Model similarity solution
(---) Reynolds stress similarity solution

CHAPTER 5

Momentum Balance Consideration

5.1 Introduction

The development of turbulence models relies heavily on experimental data in order to determine the model parameters and constants. It is therefore essential to have reliable experimental results against which the predicted values can be compared. Figures (4.1) and (4.6) show that the calculated and measured mean velocity profiles in the far field of plane and round jets are in fairly good agreement. The discrepancies are mostly at the outer edge of the jet; they are, however, significant with respect to the conservation of momentum for the axisymmetric jet since, unlike the plane jet, the largest contribution to the momentum integral comes from the region $\xi > 0.05$. This is illustrated in Figure (5.3) which shows the momentum balance for the round jet.

Based on the analysis of C. B. Baker (1980) all the measurements for the axisymmetric jet might be seriously in error since they fail to conserve momentum. He argued that, due to the high intensities of turbulence at the outer edge of the jet, the hot wire measurements are unreliable. He assumed that the measured values at the centerline of the jet, where the intensities are not as high, should be more likely to be correct.

In the present study we have considered the self-similar axisymmetric jet, and the plane jet as well. The measured mean velocity profiles are in fairly good agreement when they are normalized with their respective centerline values. (Note that an exception to this is the experiment of Abbiss et al. with a pulsed wire technique. This may be attributable to the fact that the experiment was performed for $x/d_o < 30$, where the jet may not be quite self-similar.) The major discrepancies in the measurements for both plane and round jets appear to be in the variation of the mean velocity at the centerline, see Figure (5.1) and (5.2). This might lead us to assume that the loss momentum is due to uniform errors in the measurement of the jet mean velocity throughout the flow, and at the centerline in particular, since the experimental data yield nearly the same values for the spreading rate for the respective cases, plane or round jet (see George et al. 1981).

5.2 Momentum Integral

In Appendix A the equations for the mean motion of isothermal, incompressible and self-preserving turbulent jet have been discussed in some detail. Based on order of magnitude analysis for high Reynolds number thin shear flows, the mean momentum equations have been approximated. If we retain the second order terms the integrated momentum equation across the flow reads:

$$2\pi^i \int_0^\infty [U^2 + \overline{u^2} - \frac{\overline{v^2}+\overline{w^2}i}{i+1}]y^i \, dy = \frac{M_o}{\rho} \tag{5.1}$$

where

$$M_o = \rho\pi^i \, U_o^2 \, (\frac{d_o}{i+1})^{i+1} \tag{5.2}$$

Recall that i=0 corresponds to two-dimensional and i=1 for the round jet. Equation (5.1) can be written in similarity form as:

$$\int_0^\infty [f^2 + g_1 - \frac{g_2+g_3 i}{i+1}]\xi^i \, d\xi = \frac{U_o^2}{2U_m^2} \frac{1}{\ell^{i+1}} (\frac{d_o}{i+1})^{i+1} \tag{5.3}$$

By substituting the expressions for ℓ and U_m from equation (3.26) and (3.42c), equation (5.3) takes the form:

$$\int_0^\infty [f^2 + g_1 - \frac{g_2+g_3 i}{i+1}] \, \xi^i \, d\xi = \frac{1}{2C^2} (\frac{1}{i+1})^{i+1} \tag{5.4}$$

where C is the proportionality constant in the decay law for the centerline mean velocity of the jet.

For the plane jet, the momentum integral constraint becomes

$$2C^2 \int_0^\infty [f^2 + g_1 - g_2]d\xi = 1.0 \qquad (5.5)$$

and for the round jet, we have

$$8C^2 \int_0^\infty [f^2 + g_1 - \frac{g_2+g_3}{2}]\xi d\xi = 1.0 \qquad (5.6)$$

In the equations (5.5) and (5.6) we have included the turbulent contribution to the momentum integral. The net contribution of the turbulence to the momentum amounts to about 8% of the total momentum added at the source. (see Tables (5.1) and (5.2)).

5.3 Momentum Balance

For any experimental data to be reliable it must satisfy the momentum equation (i.e., equation 5.1). This seems not to be the case in most reported experimental data.

The measurements of Wygnanski and Fiedler (1969) represent the most comprehensive attempt to characterize the axisymmetric and fully developed jet. The same can be said about the measurement of Gutmark and Wygnanski (1975) for the plane jet. For comparison with the above data, Rodi's (1975) and Abbiss et al.* (1975), measurements will be used for round jet; the data of Heskestad (1965) and Bradbury** (1965) will be used for the plane jet. The reported mean and turbulence profiles of the above experiments has been integrated graphically and the results substituted into equation (5.5) for the plane jet and in equation (5.6) for the round jet. The values of C in the above equations are either reported by the authors or have been obtained from Figures (5.1) and (5.2).

*Abbiss et al. measurements were only for $x/d_o \leq 30$ where the jet is not quite self-preserving, hence comparisons apply only to the mean velocity profiles.

**In Bradbury experiment the free stream velocity is not zero, hence the flow is strictly speaking not self-similar. (see Hinze 1975).

The results are shown in Table (5.1) and (5.2) for the round and plane jet respectively, along with the similarity solutions obtained earlier. The loss of momentum is in range 2-30% of the total momentum in the plane jet and up to 40% for the round jet.

This is sufficient to raise a serious question about the accuracy of the experimental measurements. The reported values of C are in the range 5.0-6.0 for the round and 2-2.4 for the plane jet. The discrepancy in the constant C between Rodi's data and those of Wygnanski and Fiedler, as it can be seen from Figure (5.1), is largely for $x/d_o > 60$. Note that Rodi only measured to $x/d = 75$ while Wygnanski and Fiedler measured to $x/d_o = 100$. Hence it is not quite clear whether either set is self-preserving and which is the most reliable. At the final stages of this study, S. Capp (1981) obtained $C = 6.25$ using laser Doppler anemometer in the laboratory at SUNYAB and showed that the measured velocity profile up to $x/d_o = 120$ satisfies momentum. These date were obtained too late to be analyzed further here.

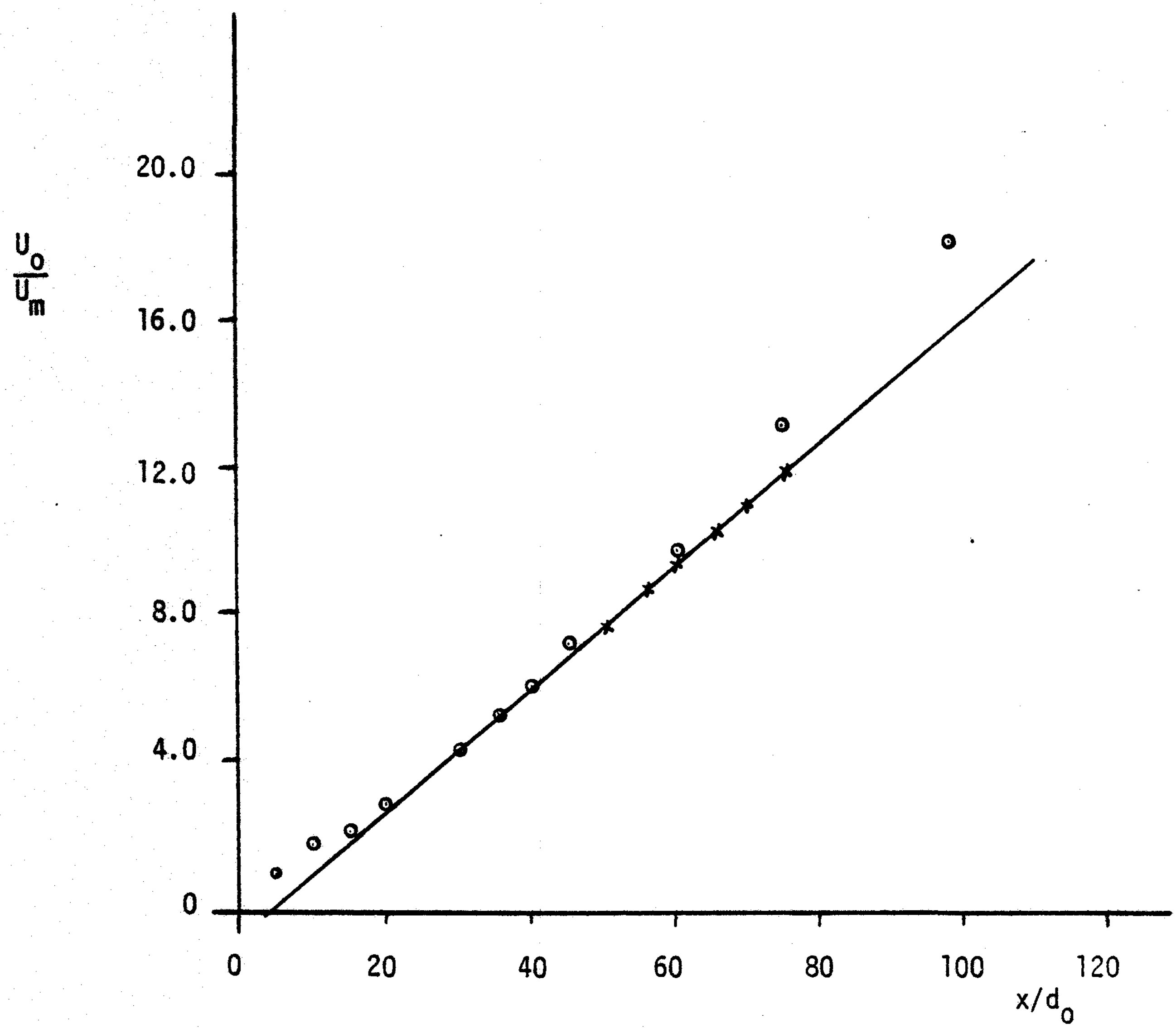

Figure 5.1 Variation of the Mean Velocity Along the Centerline of an Axisymmetric Jet.

(o) Wygnanski and Fiedler (1969).
(x) Rodi (1975).

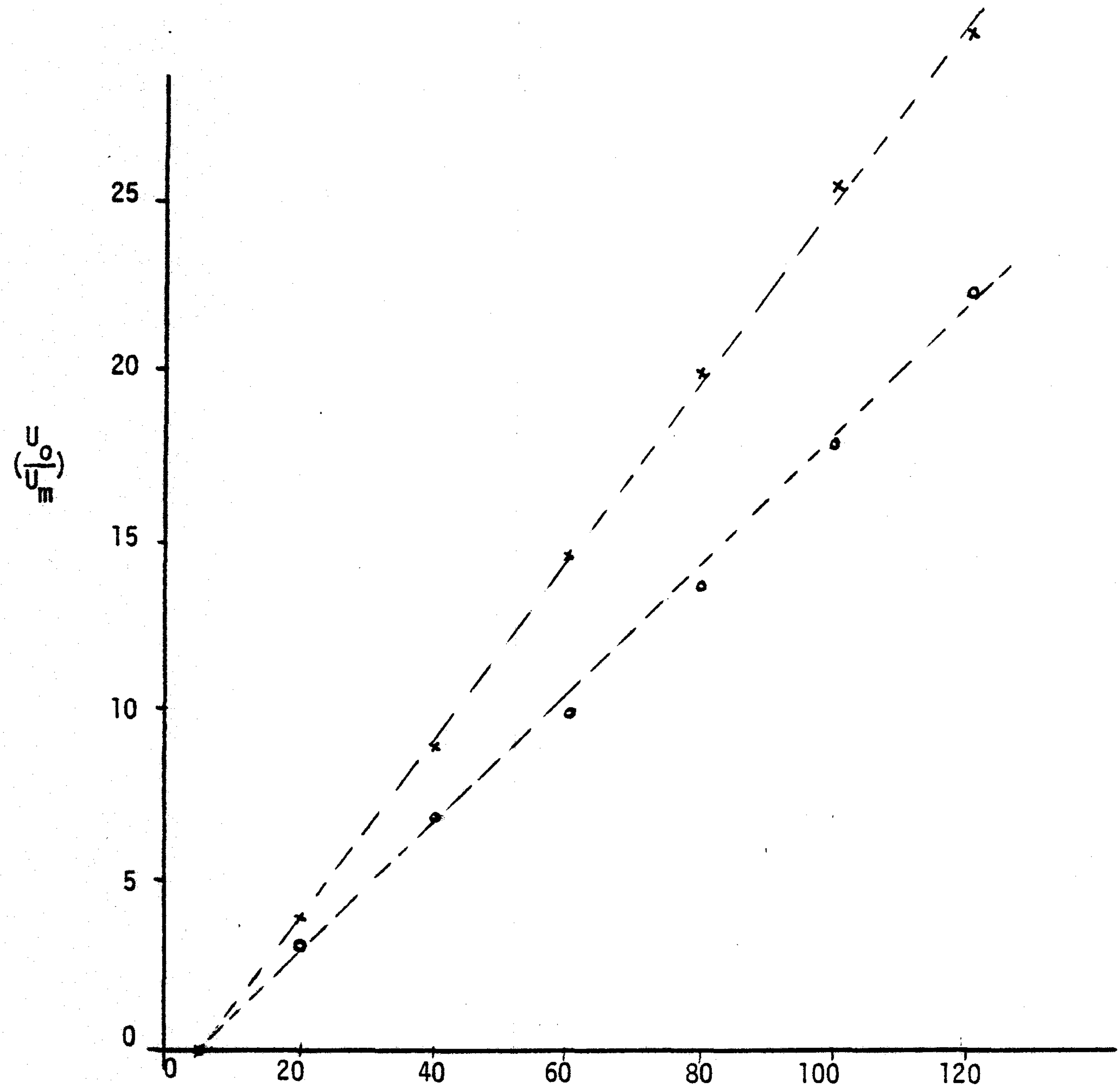

Figure 5.2 Variation of the Mean Velocity along the Centerline of the Plane Jet.

(o) Gutmark and Wygnanski (1975)
(x) Heskestad (1965)

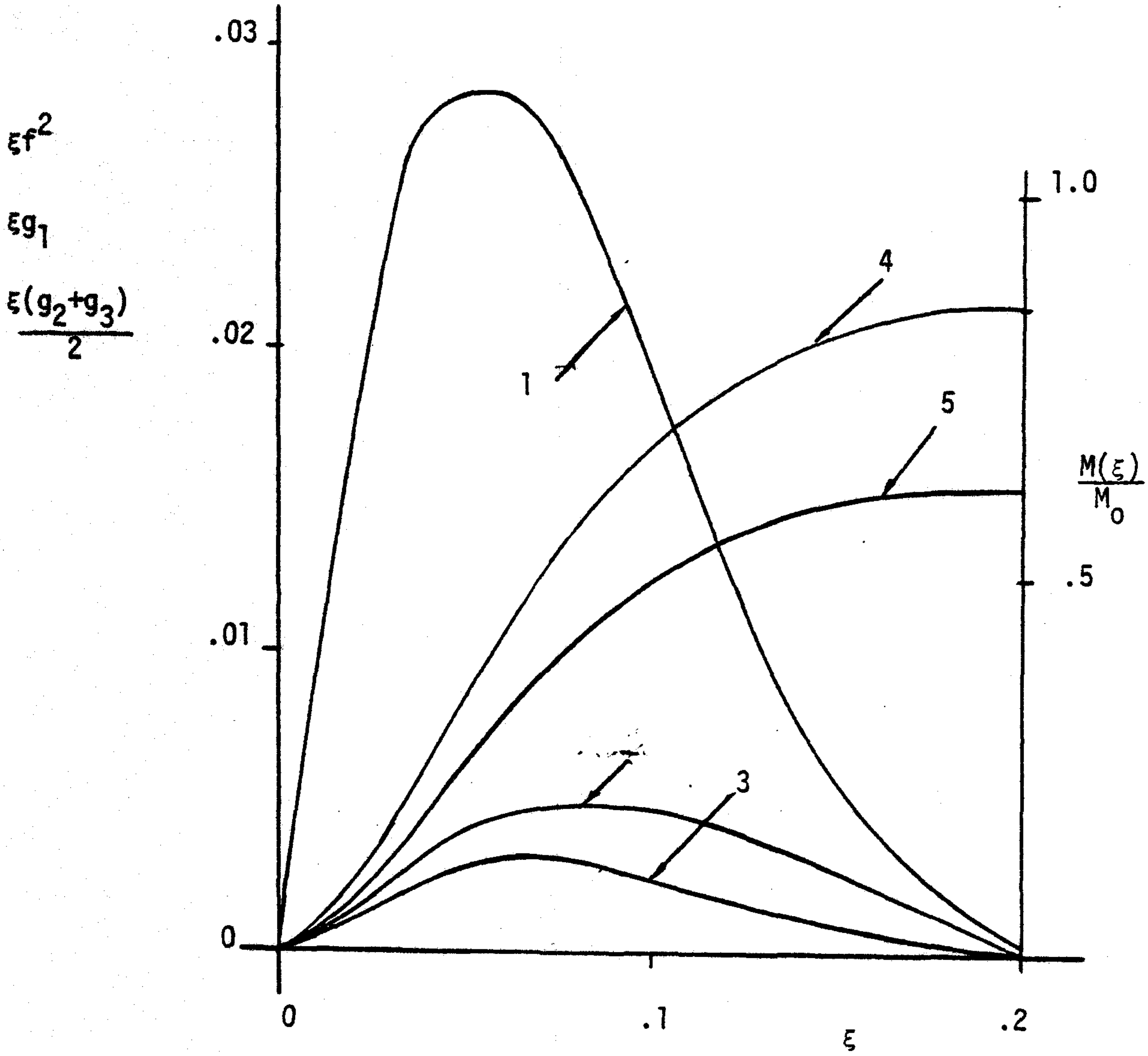

Figure 5.3 Momentum Balance for Self-preserving Axisymmetric Jet.

(1)- ξf^2, (2) - ξg_1, (3) - $\xi(g_2+g_3)/2$ from Wygnanski and Fiedler

(4)- Momentum integral evaluated from Rodi's data.

(5)- Momentum integral evaluated from Wygnanski and Fiedler's data.

Reference	$\frac{d\ell}{dx}$	C	$M_o^{-1}\int_o^\infty f^2\xi d\xi$	$M_o^{-1}\int_o^\infty g_1\xi d\xi$	$M_o^{-1}\int_o^\infty g_2\xi d\xi$	$M_o^{-1}\int_o^\infty g_3\,\xi d\xi$	$\frac{M(\infty)}{M_o}$
Wygnanski and Fiedler	.086	5.0	.540	.122	.06	.064	.60
Rodi	.086	6.0	.770	.17	.091	.106	.84
Abbiss et al.							
(laser)	.088	5.5	.75	.17	.08	-	.84
(pulsed wire)	.10		.92	.17	-	-	1.0*
Similarity Solution	.095	5.8	.919	.15	.071	.07	1.0

Table 5.1 Momentum Balance for the Axisymmetric Jet.

*Abbiss et al. did not measure $\overline{v^2}$ with the pulsed wire and the value of $\int_o^\infty \frac{\overline{v^2}+\overline{w^2}}{U_m^2}\xi d\xi$ has been assumed to be the same as in the laser measurement.

Reference	$\frac{d\ell}{dx}$	C	$M_o^{-1}\int_o^\infty f^2 d\xi$	$M_o^{-1}\int_o^\infty g_1 d\xi$	$M_o^{-1}\int_o^\infty g_2 d\xi$	$M(\infty)/M_o$
Gutmark and Wygnanski (1975)	.105	2.35	.88	.16	.06	.98
Heskestad (1965)	.112	2.0	.68	.08	.04	.72
Bradbury (1965)	.109*	2.30	.86	.10	.08	.88
Similarity Solution	.1106	2.34	.95	.1	.05	1.0

Table 5.2 Momentum Balance for Self-preserving Plane Jet.

*For Bradbury experiment U_E/U_m = .16, where U_E is the parallel free stream velocity at the outer edge of the jet, $\frac{d\ell}{dx}$ is not constant and .109 is an average value of $\frac{d\ell}{dx}$.

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CHAPTER 6

Summary and Conclusions

In the past decade considerable efforts have been given to the development of the second order models of turbulence. Much success has been achieved in predicting various classes of turbulent flows. Particularly noteworthy are the predictions of Rodi and Spalding (1970), Rodi (1972), Hanjalic and Launder (1972), Reynolds (1976), Launder and Morse (1979) and most recently Hanjalic and Launder (1980) to name but a few. In the majority of the above, second order prediction methods the model constants have been "tuned" to fit the respective experimental data (e.g. jets, wakes, mixing layers, etc.

In the present study, for that matter, the k-ε model calculation is no exception. As stated earlier the diffusion constants in the k- and ε-equations must be related so that $\sigma_\varepsilon = 2\sigma_k$. By making this modification the constants in the production/destruction terms of the dissipation rate equation, $C_{\varepsilon 1}$ and $C_{\varepsilon 2}$, have to be tuned to fit the experimental data. It has been concluded that the set of constants which predict the flow for the plane jet do not do so for the round jet. This is the best that can be done for the k-ε model at the present time. However, our main objective here in applying the k-ε model was to test several numerical schemes that have been developed for the Reynolds stress closure model. For the set of constants proposed (Table 3.2) the k-ε model predictions agree well with the experimental data. This, however, does not imply that the choice of the model constants is final, or that they are constant at all.

In the preceeding chapter (Chapter 5) it has been shown that the majority of the experimental data are in error and hence not reliable. Because of this uncertainty in the measurement there has been no attempt to "tune" the model constants in the application of the present Reynolds stress closure model to obtain better agreement with the measured values. Rather, the emphasis in the work has been to determine what the model predicts with objectively determined universal parameters and constants. In order for any model to predict unknown flows, it must be universal in nature. Hence the model parameters must be functions of the local state of turbulence which is the case in the present closure model. The return to isotropy function C_1 which also controls the diffusive transport of $\overline{u_i u_j}$ is a function of the turbulence Reynolds number and the local anisotropy.

There were wide views about the values of C_1 in previous calculations. For example, Zeman and Lumley (1976) suggest $C_1 = 3.25$. Other values suggested were 5.6 by Hanjalic and Launder (1972), 6.7 by Wyngaard and Cote (1974), 3.0 by Launder, Reece and Rodi (1975), and 2.5 by Reynolds (1976). Zeman and Tennekes (1976) obtained seven different values of C_1 between 1.8 and 3.8 by examining seven different homogeneous turbulence experiments. From the present result (Figure 4.12),* the variation of C_1 as a function of R_ℓ, II and III covers nearly all the above suggested values for this parameter. This lends credence to the view that C_1 is a universal function. The variation of ψ_0, the parameter that controls the kinetic energy decay (to be compared with $C_{\varepsilon 2}$ in Launder et al. formulation), is not too large across the jets in the present calculation or in the wake calculation of Taulbee and Lumley (1980). This suggests that the discrepancies in the Reynolds stress results of Launder and Morse (1979) are not entirely due to the effect of $C_{\varepsilon 2}$ (Launder notation), but rather to the constant used in the diffusion term in their calculations.

*The functional behavior of C_1 seems to be similar to that seen by Taulbee and Lumley (1980) in their wakes calculations.

The remaining model parameters (present model) are ψ_1, the constant in the production term of ε- equation, and c, the constant in the pressure strain terms. These were assigned fixed values. For turbulent free shear flows Reynolds (1976) suggested $\psi_1 = 2.0$ and $c = -.15$. These values appear to give reasonable results in both the round and plane jets. The lack of reliable measurement data makes the choice of ψ_1 and c difficult at the present time.

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ABSTRACT

This dissertation addresses the problem of predicting the flow field of self-preserving turbulent jets. It identifies the lack of universality of second order closure models with constant coefficients. It has been shown that the diffusion constants in the k-ε model, σ_k and σ_ε, must be related so that $\sigma_\varepsilon = 2\sigma_k$ in order to have an asymptotic solution. Based on this modification a set of constants for the (k-ε) model has been proposed.

Following Lumley (1978), a second order closure model with variable coefficients has been developed. In this formulation care is taken to satisfy realizability, non-negative quantities are never negative and Schwarz's inequality is satisfied.

An analysis of existing data for simple decaying anisotropic axi-symmetric turbulence shows that the return to isotropy function C_1 depends on the turbulent Reynolds number and the first and second invariants of the anisotropy tensor. The form proposed for C_1 by Chung (1978) is shown to violate realizability condition.

The equation for the Reynolds stresses and the dissipation rate equation are transformed to curvilinear coordinate system for the axisymmetric jet. The similarity forms of the closed Reynolds stress and dissipation rate equations along with the equations for the mean flow are solved numerically to determine the equilibrium behavior of the two-dimensional and axisymmetric jets. A numerical scheme that solves the system of equations at each grid point simultaneously is introduced. It turns out that the model with the same set of parameters and constants predicts the flow for both round and plane jets equally well.

Review of the existing measurements for the plane and round jets show that the majority of the reported experimental data are in error, since they

fail to conserve momentum. The lack of momentum balance seems to be primarily attributable to the error in the centerline measurements, and not entirely to the profile shape as eralier suspected (Baker 1980).

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TABLE OF CONTENTS

TABLE OF CONTENTS (cont.)

TABLE OF CONTENTS (cont.)

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HIGHER ORDER CLOSURE MODEL FOR TURBULENT JETS

BY

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DEDICATION

TO MY PARENTS

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TABLE OF CONTENTS

TABLE OF CONTENTS (cont.)

TABLE OF CONTENTS (cont.)

ABSTRACT

This dissertation addresses the problem of predicting the flow field of self-preserving turbulent jets. It identifies the lack of universality of second order closure models with constant coefficients. It has been shown that the diffusion constants in the k-ε model, σ_k and σ_ε, must be related so that $\sigma_\varepsilon = 2\sigma_k$ in order to have an asymptotic solution. Based on this modification a set of constants for the (k-ε) model has been proposed.

Following Lumley (1978), a second order closure model with variable coefficients has been developed. In this formulation care is taken to satisfy realizability, non-negative quantities are never negative and Schwarz's inequality is satisfied.

An analysis of existing data for simple decaying anisotropic axisymmetric turbulence shows that the return to isotropy function C_1 depends on the turbulent Reynolds number and the first and second invariants of the anisotropy tensor. The form proposed for C_1 by Chung (1978) is shown to violate realizability condition.

The equation for the Reynolds stresses and the dissipation rate equation are transformed to curvilinear coordinate system for the axisymmetric jet. The similarity forms of the closed Reynolds stress and dissipation rate equations along with the equations for the mean flow are solved numerically to determine the equilibrium behavior of the two-dimensional and axisymmetric jets. A numerical scheme that solves the system of equations at each grid point simultaneously is introduced. It turns out that the model with the same set of parameters and constants predicts the flow for both round and plane jets equally well.

Review of the existing measurements for the plane and round jets show that the majority of the reported experimental data are in error, since they

fail to conserve momentum. The lack of momentum balance seems to be primarily attributable to the error in the centerline measurements, and not entirely to the profile shape as eralier suspected (Baker 1980).

LIST OF TABLES

LIST OF FIGURES

LIST OF FIGURES (cont.)

LIST OF FIGURES (cont.)

CHAPTER I

Introduction

1.1 Background

Jet engines, wakes behind airplanes and submarines, mixing layers, water disposal in rivers, chimney plumes and all kinds of motion in the atmosphere are a few examples of turbulent free shear flows which the engineers and the meteorologists as well, wish to predict. There are, in fact, many other flows of practical importance that need not to be boundary free as in the above flows. Examples of these flows are channel, pipe, and boundary layer flows. However, the process of free turbulent mixing is prominent in all these flows. Hence the theory of free shear flows, in general, applies to these flows as well.

The above classical flows have long been favorites for turbulence investigators because of the easy manner in which they can be generated in the laboratory. Another important characteristic of these flows, is their tendency to become fully developed and self-preserving (at least in principle) after a certain development region. This enables theoretical investigators to approximate the equations of motion based on physical grounds,such as order of magnitude analysis.

At the turn of the century the advances in the study of turbulent-flow problems were made primarily in the laboratory where basic insights into the general nature of turbulent flows were developed and the behavior of selected families of flows were studied systematically. For engineers and meteorologists there were only a limited number of useful tools, such as boundary layer prediction methods which solve the momentum integral equation with a high empirical content. Turbulent flow features such as sudden changes in boundary conditions, separation or recirculation could not be predicted by these early methods with any degree of reliability. Hence empirical work remained an essential ingredient in many engineering analysis.

Halfway into this century the computer began to have a major impact on engineering computations and the development of a theoretical model capable of predicting turbulent flows with a fair degree of accuracy began to attract many researchers in this field.

The exact equations that govern turbulent flows are well known; they are the Navier-Stokes equations. These equations, which are accepted as the fundamental basis for turbulent flow problems, are non-linear and strongly coupled; hence, an analytical approach leading to closed form solutions is not possible. Procedures exist to solve these equations numerically. However, the energy-dissipating eddies are so small that the computational mesh required must be so fine that realistic calculations cannot be carried out with present day computer hardware. Therefore it is customary to consider statistical properties of turbulence, which is often sufficient in providing engineers with the required information. This approach, however, leads to an infinite number of correlation equations that govern the turbulence properties.

A practical way to close the system of equations is to employ a turbulent model which approximates higher order correlations (moments) in terms of lower order moments that can be calculated. This approximation relies heavily on experimental data to determine the model empirical constants and functions. Therefore a reliable set of experimental data must be provided to serve as a basis for any theoretical prediction methods.

1.2 Theoretical Model

The turbulence models are classified either according to the number of partial differential equations they employ for turbulent quantities or by the order of the moment for which the transport equations are written.

The first turbulence model which has been applied to turbulent free shear flows with some success, is Prandtl's (1925) mixing-length hypothesis. This simple model relates the turbulent shear stress uniquely to the local mean velocity gradient. Then the partial differential equation for the mean flow is transformed to ordinary differential equations for which an analytical solution can be obtained. (see i.e. Appendix B) This model, among others of its class, often breaks down in many situations when there is more than one mechanism present, producing, in general, more than one length or velocity scale.

A second order model is expected to work better in most situations because it carries transport equations for second order quantities, so that many of the mechanisms responsible for the production of those quantities are represented accurately. Kolmogorov (1942), Prandtl (1945), Chou (1945) and Rotta (1951) laid the foundation for second order models of turbulence. However, analytical solutions for the resulting system of equations could not be obtained and a numerical one was not possible at that time.

By the early 70's when advances in computers and numerical methods overcame the mathematical difficulties, several predictions of turbulent free shear flows had been made with a fair degree of accuracy. Among the reported models are the (k-ε) model proposed by Jones and Launder (1972), (k-$k\ell$) model by Rodi and Spalding (1971) and the (k-ω) model by Spalding (1972). However, these prediction methods use model constants which were thought to be universal, but the calculations showed that they are not. For example, a set of constants that predict the flow for plane jets will not do so for the round jet.

Furthermore the two equation model used the eddy viscosity concept (e.g. $\nu_t \sim k^2/\varepsilon$), hence they do not keep track of the dynamics of all the second order correlations of importance. This led to the idea (Donaldson, 1971; Hanjalic and Launder, 1972b; Bradshaw, 1972) that the most promising class of turbulence models for making numerical calculations of such complex flows is that based on the solution of the approximated equations for the Reynolds stresses $\overline{u_i u_j}$ and indeed several proposals have been made (see section 2.4).

1.3 Scope and Object

In the past decade considerable success (Lumley and Khajeh-Nouri, 1974; Launder, Reece and Rodi, 1975; Reynolds, 1976 and Hanjalic and Launder 1976) have been made in predicting shear layer, jet wakes, channel flows, and boundary layers with reasonable degree of accuracy. There were, however, some unexplained differences between calculated and measured turbulent quantities.

These discrepancies arise from the neglect of some correlation terms in the governing equations, incomplete or inappropriate closure formulations for other correlations or simply not having the optimum values for the coefficients in closure formulations which may be functionally correct. For instance a set of constants in the closure formulations that gives good results for one flow situation sometimes does not work well for another flow. This is the case with the predictions for the two-dimensional and round jet flows (Launder and Morse, 1979).

Although some fundamental guiding principles, i.e. invarient modeling, have been used in formulating closure relations, much is developed by *ad hoc* assumptions. With appropriately adjusted constants some of these *ad hoc* closures have performed admirably well. However, one would like to develop closure formulations from first principles using rational procedures. Also it would be highly desirable that the model parameters and constants be determined as part of the calculation, or at least, determined from certain "key" basic experiments. Furthermore, closure formulations and the resulting theory should not violate certain mathematical or physical principles, e.g. conservation of mass and momentum.

Using a rational approach, Lumley (1978) formulated a second order model that is an orderly expansion about a homogeneous, stationary turbulence, the large scales of which have a Gaussian distribution. In this formulation care is taken to satisfy realizability conditions. This condition implies that non-negative quantities are never negative and Schwarz's inequality is satisfied. The key coefficients in this closure relation are functions of the local turbulent Reynolds number and anisotropy.

The primary aim of this dissertation is to consider the above closure formulation and investigate the functional form of the model parameters based on the available data for a homogeneous decaying axisymmetric turbulent flow. The closed Reynolds stress and dissipation equations are transformed to curvilinear coordinates for the use in the axisymmetric jet calculations.

The similarity forms of the resulting system of equations for plane and axisymmetric flow are solved numerically to determine the equilibrium behavior of turbulent (isothermal) fully developed and self-similar jets. The results are compared with available experimental data with the emphasis on conservation of momentum.

C. B. Baker (1980) raised the question about the validity of the axisymmetric jet measurements, since they failed to conserve momentum. He analyzed the data of Wygnanski and Fiedler (1969) for an axisymmetric self-preserving jet and argued that the measured mean velocity profile conserves only half of the momentum added at the source. (See also George, Seif and Baker, 1981).

On the other hand the most recently measured and calculated profiles are fairly in good agreement with Wygnanski and Fiedler profiles when they are normalized with their respective centerline value of the mean velocity. Hence part of this study (chapter 5) is devoted to examination of the jet data (plane and axisymmetric) in contrast with the results of theoretical predictions.

CHAPTER 2

The Reynolds Stress Closure

2.1 Equations for the Mean Flow

The equations that govern the mean motion of an incompressible isothermal turbulent flow are obtained from the Navier-Stokes equations. By decomposing the instantaneous velocity and pressure into a mean and turbulent component and by taking the time average of all terms, the following equations will result (see Tennekes and Lumley 1972).

Conservation of mass:

$$U_{i,i} = 0 \tag{2.1}$$

Conservation of momentum:

$$\rho\dot{U}_i + \rho U_j U_{i,j} = -P_{,i} + (\mu U_{i,j} - \rho\overline{u_i u_j})_{,j} \tag{2.2}$$

where the overbar denotes the time average, the overdot stands for the partial derivative with respect to time,and the subscripts after the commas denote the partial differentiation, eg. $U_{i,j} = \frac{\partial U_i}{\partial x_j}$. The new unknown $\rho\overline{u_i u_j}$ in the momentum equation is the contribution of the turbulent motion to the mean stress tensor. It is known as the Reynolds stress in honor of Reynolds who first developed equation (2.2) in (1785). The Reynolds stress $\rho\overline{u_i u_j}$ has nine components and hence introduces nine unknowns to the equation of motion; however,since it is a symmetric tensor ($\overline{u_i u_j} = \overline{u_j u_i}$) the number of unknowns is reduced to six, three normal and three tangential components.

The aim of any prediction method in turbulent modeling is to solve the momentum equation for U_i,but because of the presence of $\overline{u_i u_j}$ in the momentum equation, the system of equations in (2.1) and (2.2) do not constitute a closed set. Closing this set of equations has been of major concern for over a century. An earlier closure, which is known today as the zero order model, was originally proposed by Boussinesq in 1877. This simple closure model assumes that the shear stress is proportional to the mean velocity gradient. This approximation predicts the velocity and shear stress profiles for the self preserving turbulent jet (see Appendix C) with a good degree of accuracy over most of the flow region, but it fails to do so when the turbulence is in non-self-preserving state. However, with the advances of electronic facilities researchers have tried to develop a universal method to predict the Reynolds stress accurately. The most direct way to determine $\overline{u_i u_j}$, of course, is to solve a transport equation for all non-zero components of the Reynolds stress. Such an equation, in fact, does exist and it will be discussed in the following section.

2.2 The Reynolds Stress Equation

A transport equation that governs the Reynolds stress can be derived in the following way. The equation for the component i of the instantaneous velocity (U_i+u_i) is multiplied by u_j and the equation for the j component $U_j+u_j)$ is multiplied by u_i. Summing of the two equations and taking the time average yields the desired equation for $\overline{u_i u_j}$ (see Hinze,1959):

$$\dot{\overline{u_i u_j}} + U_k(\overline{u_i u_j})_{,k} = - (\overline{u_i u_k}\, U_{j,k} + \overline{u_j u_k}\, U_{i,k}) + \overline{\frac{p}{\rho}\,(u_{j,i} + u_{i,j})}$$

(i) = convection (ii) = production (iii) = pressure strain

$$- [\overline{u_i u_j u_k} - \nu(\overline{u_i u_j})_{,k} + \overline{\frac{p}{\rho}\,(u_i \delta_{jk} + u_j \delta_{ik})}]_{,k}$$

(iv) = diffusion

$$- 2\, \nu\, \overline{u_{i,k} u_{j,k}} \qquad (2.3)$$

(v) = dissipation

As it can be seen from equation (2.3), further unknowns, such as the triple correlation and pressure velocity correlations, have been introduced. This, of course, adds to the complexity of the situation. Transport equations for the third order statistical moment $\overline{u_i u_j u_k}$ can be again derived in a way similar to the above; however, the number of unknowns will grow faster than the number of equations. Closing the system in equation (2.3) at the second order level (the Reynolds stress closure) will be discussed later in this chapter.

2.3 The Kinetic Energy Equation

For future reference let us take a look at the turbulent kinetic energy equation. Contraction of equation (2.3) leads to an important equation, the kinetic energy equation of the turbulence:

$$\underset{(i)}{\dot{\overline{q^2}} + U_j \overline{q^2}_{,j}} = \underset{(ii)}{-[\overline{u_j(q^2 + 2\tfrac{p}{\rho})}]_{,j}} - \underset{(iii)}{2\,\overline{u_i u_j}\,U_{i,j}} + \underset{(iv)}{2\,\nu(\overline{u_i u_{i,j}})_{,j}}$$

$$- \underset{(v)}{2\nu\,\overline{u_{i,j} u_{i,j}}} \qquad (2.4)$$

where $\overline{q^2} = \overline{u_i u_i}$.

Equation (2.4) states: The change in (i), the turbulent kinetic energy per unit mass and unit time including the convection transport by the mean motion, is equal to (ii), the convective diffusion by the turbulence of the total mechanical energy or the work by the total dynamic pressure of the turbulence, plus (iii), the work of deformation of the mean motion by the turbulence stresses, plus (iv), the work done by the viscous stresses of the turbulent motion, plus (v), the viscous dissipation by turbulent motion. To close the system of equations (2.2) and (2.4) at this level, which is known in the literature as the one equation model, the terms on the right hand side of equation (2.4) will be approximated employing the eddy viscosity concept and specifying a characteristic length scale (see Reynolds 1976). However, if we go one step further and derive an additional transport equation for the dissipation ε, then we have the so called two equation model. This model eliminates the need for specifying a characteristic length scale as function of position throughout the flow by defining the eddy viscosity as $\nu_t \sim \overline{q^2}^2/\varepsilon$. A detailed discussion of this model and its application will be presented in Chapter three.

2.4 The Dissipation Rate Equation

An exact transport equation for the dissipation rate of turbulent kinetic energy (i.e., the correlation $\nu\,\overline{u_{i,\ell}u_{i,\ell}}$) can be developed from the Navier-Stokes equations for the fluctuating velocities by appropriate differentiation, multiplication and averaging. The resulting equation can be written as (see Daly & Harlow 1970):

$$\dot{\varepsilon} + U_k\varepsilon_{,k} = \underset{\text{(i)}}{-2\,\nu\,\overline{u_{i,k}u_{i,j}u_{k,j}}} - \underset{\text{(ii)}}{2\,\overline{(\nu u_{i,jk})^2}}$$

$$\underset{\text{(iii)}}{-(\overline{u_k\varepsilon} + 2\,\frac{\nu}{\rho}\,\overline{u_{k,i}P_{,j}} - \nu\varepsilon_{,k})_{,k}}$$

$$\underset{\text{(iv)}}{-2\,\nu(\overline{u_{i,j}u_{k,j}} + \overline{u_{j,i}u_{j,k}})\,U_{i,k}}$$

$$\underset{\text{(v)}}{-2\,\nu\overline{u_k u_{i,j}}\,U_{i,jk}} \tag{2.5}$$

It is an extremely difficult task to consider equation (2.5) in its entirety. Luckily for high Reynolds numbers flow (i.e. most of the turbulent flows) a great simplification will result when an order of magnitude analysis is employed (Tennekes and Lumley 1972). The terms (i) and (ii) which represent the generation of ε by the stretching of vortex filaments, and the destruction through the tendency of viscosity to reduce the instantaneous velocity gradients are the most dominant terms. However their difference, which really matters, is nearly the same order of magnitude as the transport terms (iii). The terms (iv) and (v) are smaller than other terms by at least a factor of $R_\ell^{1/2}$, where R_ℓ is the turbulent Reynolds number; therefore, these terms can be safely ignored. Hence equation (2.5) can be written as

$$\dot{\varepsilon} + U_k \varepsilon_{,k} = - \underbrace{(\overline{u_k \varepsilon} + 2 \frac{\nu}{\rho} \overline{P_{,i} u_{k,i}})_{,k}}_{\text{(i)}} - \underbrace{2 \nu \overline{u_{i,k} u_{i,j} u_{k,j}}}_{\text{(ii)}} - \underbrace{2 \overline{(\nu u_{i,jk})^2}}_{\text{(iii)}} \qquad (2.6)$$

Still the terms on the right hand side of equation (2.6) add further unknowns into the equation set governing the Reynolds stress. These terms are not directly accessible to measurement and therefore their approximation can be only verified indirectly by determining whether the predicted distribution of ε is consistent with the measured variation of the turbulent kinetic energy through a particular shear flow. Modeling of the transport terms, (i), and production-destruction terms, (ii) and (iii), in equation (2.6) will be included in the analysis of the Reynolds stress closure.

2.5 The Reynolds Stress Closure Approximation

The Reynolds stress model begins with the equations (2.1), (2.2), (2.3) and (2.6). In order to solve equation (2.3) for $\overline{u_i u_j}$, some information about the higher order moments $\overline{u_i u_j u_k}$ and $\overline{P u_{i,j}}$ must be provided. Those terms will be approximated as functions of the lower order moments. Such approximations will rely heavily on experimental data to determine the proportionality constants and certain key parameters.

The idea of proposing a model like (2.3) was first suggested by Rotta (1951). Some predictions have been recently made following this idea by Daly and Harlow (1970), Reynolds (1970) Donaldson (1971), Noat, Shavit and Wolfstein (1972), Hanjalic and Launder (1972) and Lumley and Khajah-Nouri (1974), to name a few. However there have been widely different views on how to treat the third order moments, the triple velocity correlation in particular. Before we proceed with the analysis of closing the Reynolds stress equations, a new arrangement of the terms involved in equation (2.3) will be made. For convenience in later analysis we will separate the effects of the various terms to be modeled and group them according to their rules and functions in the equations of motion.

An expression for the pressure can be obtained by taking the divergence of the Navier-Stokes equations for the fluctuating velocity component u_i. The result is (Lumley, 1978)

$$-\frac{p_{,ii}}{\rho} = 2\,U_{i,j}u_{j,i} + \overline{u_{i,j}u_{j,i}} - \overline{u_i u_{j,ij}} \qquad (2.7)$$

The right hand side of equation (2.7) contains two types of terms. The first term is linear in the fluctuating velocity and related to the mean velocity gradients while the second and third are nonlinear in the fluctuating velocity. If we conveniently split the pressure such that

$$-\frac{p^{(1)}_{,ii}}{\rho} = 2\,U_{i,j}u_{j,i} \qquad (2.8)$$

$$-\frac{p^{(2)}_{,ii}}{\rho} = \overline{u_{i,j}u_{j,i}} - \overline{u_i u_{j,ij}} \qquad (2.9)$$

where the correlations with $p^{(1)}$ and its gradients are known as the "rapid terms". While the correlations with $p^{(2)}$ are known as the

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APPENDICES

APPENDIX A

Equations of Motion for the Mean Flow

The dynamic equations that describe the mean motion for turbulent free jets are obtained from Navier-Stokes equations (Section 2.1). Equation (2.1) and (2.2) are written in cartesian tensor notation. In this appendix the equations for the mean flow will be presented in component form for the plane and axisymmetric turbulent free jets. An order of magnitude analysis will be performed based on physical grounds. By integrating the mean momentum equation and by using the continuity equation to eliminate the cross stream mean velocity the momentum integral constraint will result.

I. The Plane Jet

For two dimensional turbulent free jet of an incompressible isothermal fluid, issuing in still surrounding, the equations for the mean motion are obtained from equations (2.1) and (2.2). For steady motion ($\frac{\partial}{\partial t} = 0$) the equation can be written in cartesian components as follows:

Continuity:

$$\frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} + \frac{\partial W}{\partial z} = 0 \tag{A1}$$

x-momentum:

$$U\frac{\partial U}{\partial x} + V\frac{\partial U}{\partial y} + W\frac{\partial U}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\nabla^2 U - \frac{\partial \overline{u^2}}{\partial x} - \frac{\partial \overline{uv}}{\partial y} - \frac{\partial \overline{uw}}{\partial x} \tag{A2}$$

y-momentum:

$$U\frac{\partial V}{\partial x} + V\frac{\partial V}{\partial y} + W\frac{\partial V}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu\nabla^2 V - \frac{\partial \overline{uv}}{\partial x} - \frac{\partial \overline{v^2}}{\partial y} - \frac{\partial \overline{vw}}{\partial z} \tag{A3}$$

z-momentum:

$$U\frac{\partial W}{\partial x} + V\frac{\partial W}{\partial y} + W\frac{\partial W}{\partial z} = -\frac{1}{\rho}\frac{\partial P}{\partial z} + \nu\nabla^2 W - \frac{\partial \overline{wu}}{\partial x} - \frac{\partial \overline{wv}}{\partial y} - \frac{\partial \overline{w^2}}{\partial z} \tag{A4}$$

For the plane jet the following assumptions are made;

i) There is no mean motion in the z-direction.

ii) All derivatives with respect to this z-coordinate are zero.

iii) The shear stress $\overline{uw}$ and $\overline{vw}$ are zero.

Applying the above assumption equation (A4) becomes

$$\frac{\partial}{\partial z}(P - \rho\overline{w^2}) = 0$$

which states that $P - \rho\overline{w^2}$ is a function of x and y only. The equations (A1-A3) reduce to;

$$\frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} = 0 \qquad \text{(A6)}$$

$$U\frac{\partial U}{\partial x} + V\frac{\partial U}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\nabla^2 U - \frac{\partial\overline{u^2}}{\partial x} - \frac{\partial\overline{uv}}{\partial y} \qquad \text{(A7)}$$

$$U\frac{\partial V}{\partial x} + V\frac{\partial V}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial v} + \nu\nabla^2 V - \frac{\partial\overline{uv}}{\partial x} - \frac{\partial\overline{v^2}}{\partial y} \qquad \text{(A8)}$$

Obviously the principal mean velocity in the jet flow is the axial component U and hence the x-momentum is the principal equation of motion for the mean flow. However, we have to examine the cross-stream momentum equation (y-momentum) and analyze each term based on its order of magnitude relative to the leading terms.

Let us consider the region far away from the jet exit, i.e., when the mean flow becomes almost parallel and the boundary layer approximation are applicable. In the far field of the flow we can identify two velocity scales and two length scales. (See Figure (A1)). Let L and ℓ be length scales in the x- and y-direction respectively and let U_m be the mean velocity scale and u is some characteristic turbulent velocity scale such that;

$$x \approx o(L)\ ,\ y \approx o(\ell)\ ,\ \overline{u^2} \simeq \overline{v^2} \simeq \overline{w^2} \simeq o(u^2)\ ,\ U = o\ (U_m)$$

and from conservation of mass $V \simeq U_m(\frac{\ell}{L})$.

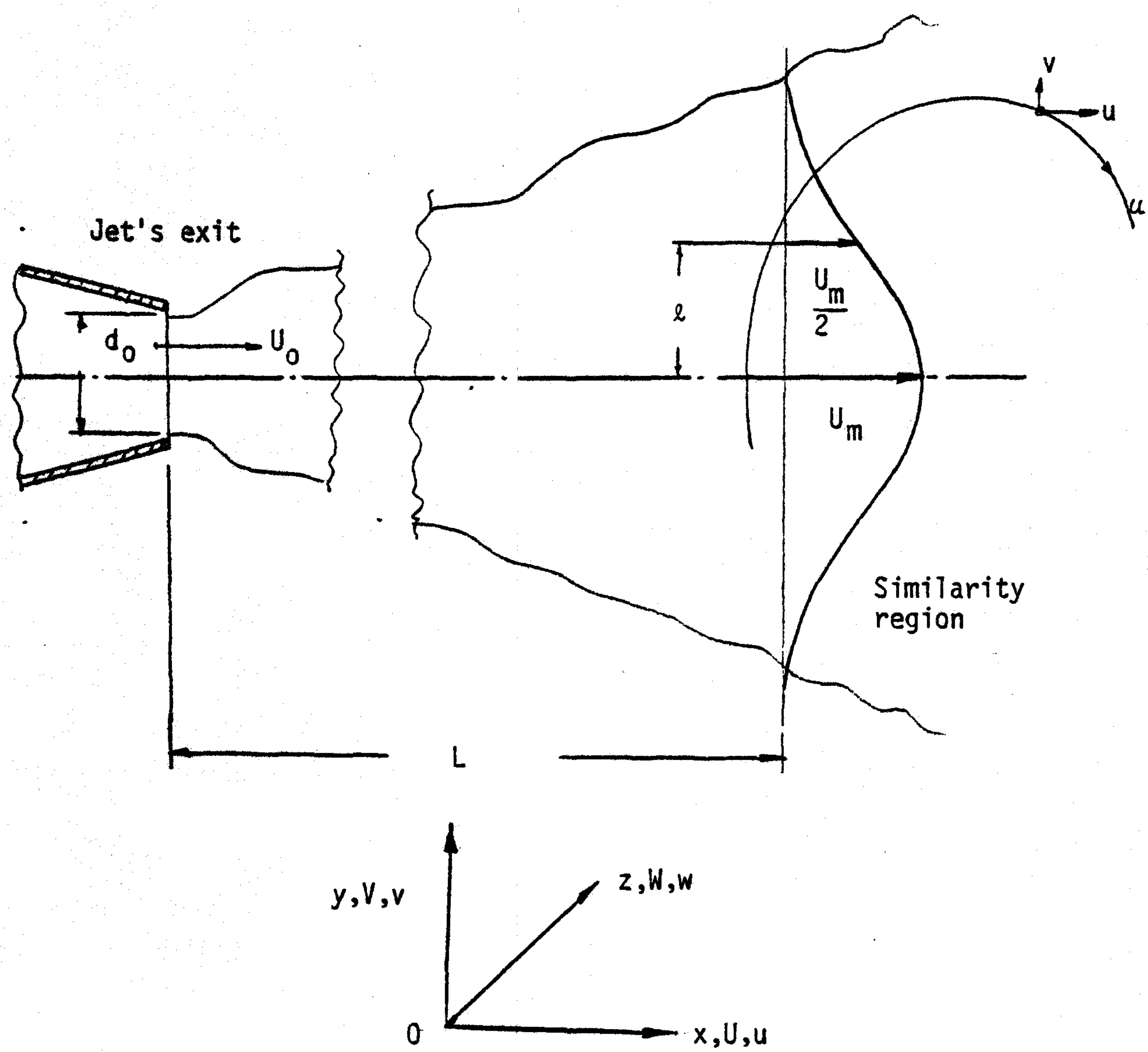

Figure A1. Schematic Diagram of the Two Dimensional Jet and Coordinate System.

Where the "o" stands for order of magnitude. Hence the terms in the y-momentum can be scaled as follows:

$$U\frac{\partial V}{\partial x} + V\frac{\partial V}{\partial y} = \frac{1}{\rho}\frac{\partial P}{\partial y} + \nu\left(\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2}\right) - \frac{\partial \overline{uv}}{\partial x} - \frac{\partial \overline{v^2}}{\partial y}$$

$$\left(\frac{\ell}{L}\right)^2 \frac{U_m^2}{\ell} + \left(\frac{\ell}{L}\right)^2 \frac{U_m^2}{\ell} = (\ ?\ ) + \frac{\nu}{U_m \ell}\left[\left(\frac{\ell}{L}\right)^2 + \left(\frac{\ell}{L}\right)\right]\frac{U_m^2}{\ell} - \left(\frac{u}{U_m}\right)^2\left(\frac{\ell}{L}\right)\frac{U_m}{\ell} - \left(\frac{u}{U_m}\right)^2 \frac{U_m^2}{\ell} \tag{A9}$$

In most free turbulent shear flows where the turbulent Reynolds number $R_\ell = \frac{U_m \ell}{\nu}$ is relatively high we have;

$$\left(\frac{u}{U_m}\right)^2 - \left(\frac{\ell}{L}\right) << 1.0 \tag{A10}$$

and

$$\frac{1}{R_\ell} << 1.0 \tag{A11}$$

Hence by neglecting higher order terms in $\left(\frac{u}{U_m}\right)^2$ and $\left(\frac{\ell}{L}\right)$ the terms that will be retained in equation (A8) are the first terms on the right side of the equation because nothing can be said about this term except that it must be of the same order of the last term in the equation. Therefore we must have,

$$-\frac{1}{\rho}\frac{\partial p}{\partial y} = \frac{\partial \overline{v^2}}{\partial y} \tag{A12}$$

If we integrate equation (A12) with respect to y, differentiate the resulting equation with respect to x and substitute into equation (A7) the pressure is eliminated and the momentum equation becomes:

$$U\frac{\partial U}{\partial x} + V\frac{\partial U}{\partial y} = -\frac{\partial}{\partial x}\left(\overline{u^2} - \overline{v^2}\right) - \frac{\partial \overline{uv}}{\partial y} + \nu\left(\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2}\right) \quad \text{A13)}$$

Using order of magnitude analysis similar to the above it can be shown that the first and last terms on the right side of equation (A13) for first order approximation are negligibly small compared to the rest of the terms for the flow considered. So, the momentum equation can be approximated as:

$$U\frac{\partial U}{\partial x} + V\frac{\partial U}{\partial y} = - \frac{\partial(\overline{uv})}{\partial y} \tag{A14}$$

Momentum Integral Constraint

Assuming that we retain all the terms in equation (A13) but the smallest term $(\frac{\partial^2 U}{\partial x^2})$ then we have,

$$U\frac{\partial U}{\partial x} + V\frac{\partial U}{\partial y} + \frac{\partial}{\partial x}(\overline{u^2} - \overline{v^2}) + \frac{\partial \overline{uv}}{\partial y} = \nu\frac{\partial^2 U}{\partial y^2} \tag{A15}$$

if we make use of the continuity equation the momentum equation becomes;

$$\frac{\partial U^2}{\partial x} + \frac{\partial UV}{\partial y} + \frac{\partial}{\partial x}(\overline{u^2} - \overline{v^2}) + \frac{\partial \overline{uv}}{\partial y} = \nu\frac{\partial^2 U}{\partial y^2} \tag{A16}$$

We integrate equation (A16) across the jet to obtain;

$$2\int_0^\infty \frac{\partial}{\partial x}[U^2 + \overline{u^2} - \overline{v^2}]\,dy = -\,2\overline{uv}\Big|_0^\infty - 2UV\Big|_0^\infty + 2\nu\frac{\partial U}{\partial y}\Big|_0^\infty$$

Since the terms on the right side of the equation vanish at both limits, the momentum integral becomes,

$$\frac{d}{dx}\int_0^\infty (U^2 + \overline{u^2} - \overline{v^2})dy = o \tag{A17}$$

Integration with respect to x leads to:

$$2\int_0^\infty (U^2 + \overline{u^2} - \overline{v^2})dy = M_o/\rho \tag{A18}$$

where M_o is the momentum added at the source and defined by

$$M_o = \rho U_o^2 d_o \tag{A19}$$

II. The Axisymmetric Jet

The equations that govern the mean motion of an axisymmetric incompressible and isothermal turbulent flow are given by

Continuity:

$$\frac{1}{r}\frac{\partial}{\partial r}(rV) + \frac{1}{r}\frac{\partial W}{\partial \theta} + \frac{\partial U}{\partial x} = 0 \qquad \text{(A20)}$$

r-momentum:

$$\frac{dV}{dt} - \frac{W}{r} = -\frac{1}{\rho}\frac{\partial p}{\partial r} + \nu(\nabla^2 V - \frac{V}{r^2} - \frac{2}{r^2}\frac{\partial W}{\partial \theta}) - \frac{1}{r}\frac{\partial}{\partial r}(r\overline{v^2})$$

$$-\frac{1}{r}\frac{\partial}{\partial \theta}(\overline{vw}) - \frac{\partial}{\partial x}(\overline{uv}) + \frac{\overline{w^2}}{r} \qquad \text{(A21)}$$

θ-momentum:

$$\frac{dW}{dt} - \frac{WV}{r} = -\frac{1}{\rho}\frac{1}{r}\frac{\partial p}{\partial \theta} + \nu(\nabla^2 W - \frac{W}{r^2} + \frac{2}{r^2}\frac{\partial V}{\partial \theta}) - \frac{1}{r}\frac{\partial}{\partial \theta}\overline{w^2}$$

$$-\frac{\partial}{\partial r}(\overline{wv}) - \frac{\partial}{\partial x}(\overline{wu}) - 2\frac{\overline{wv}}{r} \qquad \text{(A22)}$$

x-momentum:

$$\frac{dU}{dt} = -\frac{1}{\rho}\frac{\partial P}{\partial x} + \nu\nabla^2 U - \frac{\partial \overline{u^2}}{\partial x} - \frac{1}{r}\frac{\partial}{\partial r}(r\overline{uv}) - \frac{1}{r}\frac{\partial}{\partial \theta}(\overline{uw}) \qquad \text{(A23)}$$

where

$$\frac{d}{dt} = \frac{\partial}{\partial t} + V\frac{\partial}{\partial r} + \frac{W}{r}\frac{\partial}{\partial \theta} + U\frac{\partial}{\partial x} \qquad \text{(A24)}$$

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial x^2} \qquad \text{(A25)}$$

For the round turbulent jet with no swirl the symmetry requires that the azimuthal component of the mean velocity W and the shear stress $\overline{uw}$ and $\overline{wv}$ are zeros and all derivatives with respect to the azimuthal coordinate ϕ are identically zero. Hence the system of equations (A20)-(A23) for steady motion ($\frac{\partial}{\partial t} = 0$) becomes

$$\frac{1}{r}\frac{\partial}{\partial r}(rV) + \frac{\partial U}{\partial x} = 0 \tag{A26}$$

$$V\frac{\partial V}{\partial r} + U\frac{\partial V}{\partial x} = -\frac{1}{\rho}\frac{\partial p}{\partial r} + \nu(\nabla^2 V - \frac{V}{r^2}) - \frac{1}{r}\frac{\partial}{\partial r}(r\overline{v^2}) - \frac{\partial}{\partial x}\overline{uv} + \frac{\overline{w^2}}{r} \tag{A27}$$

$$V\frac{\partial U}{\partial r} + U\frac{\partial U}{\partial x} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\nabla^2 U - \frac{\partial \overline{u^2}}{\partial x} - \frac{1}{r}\frac{\partial}{\partial r}(r\overline{uv}) \tag{A28}$$

Using the same velocity and length scales as in the two dimensional case replacing r by y, the terms in the r-momentum equation (A27) will be approximated as follows:

$$V\frac{\partial V}{\partial r} + U\frac{\partial V}{\partial x} = -\frac{1}{\rho}\frac{\partial p}{\partial r} + \nu(\frac{\partial^2 V}{\partial r^2} + \frac{1}{r}\frac{\partial V}{\partial r} + \frac{\partial^2 V}{\partial x} - \frac{V}{r^2}) - \frac{1}{r}\frac{\partial}{\partial r}(r\overline{v^2})$$

$$(\frac{\ell}{L})^2\frac{U_m^2}{\ell} + (\frac{\ell}{L})^2\frac{U_m^2}{\ell} = (\ ?\ ) + \frac{2}{U_m\ell}(\frac{\ell}{L} + \frac{\ell}{L} + (\frac{\ell}{L})^3 - \frac{\ell}{L})\frac{U_m}{\ell} - (\frac{u}{U_m})^2\frac{U_m^2}{\ell}$$

$$-\frac{\partial\overline{uv}}{\partial x} \qquad + \qquad \frac{\overline{w^2}}{r}$$

$$(\frac{u}{U_m})^2\frac{\ell}{L}\frac{U_m^2}{\ell} + (\frac{u}{U_m})^2\frac{U_m^2}{\ell} \tag{A29}$$

Again, if the second order terms are ignored, i.e. terms of order $(\frac{\ell}{L})^2$ or $(\frac{\ell}{L})(\frac{u^2}{U_m^2})$ and when $R\ell = \frac{U_m\ell}{\nu}$ is sufficiently large enough, all that is left in the r-equation is,

$$-\frac{1}{\rho}\frac{\partial p}{\partial r} + \frac{1}{r}\frac{\partial}{\partial r}(r\overline{v^2}) + \frac{\overline{w^2}}{r} = 0 \tag{A30}$$

Now integrate equation (A29) from same reference r to ∞.

$$-\frac{p_\infty - p}{\rho} = \int_r^\infty \frac{1}{r}\frac{\partial}{\partial r}(r\overline{v^2})dr - \int_r^\infty \frac{\overline{w^2}}{r}dr$$

The first term on the right can be integrated by parts so that

$$- \frac{P_\infty - p}{\rho} = - \overline{v^2} + \int_0^\infty \frac{\overline{v^2}-\overline{w^2}}{r}\, dr \tag{A31}$$

Differentiating equation (A31) with respect to x with P_∞ = const we have,

$$- \frac{1}{\rho}\frac{\partial p}{\partial x} = \frac{\partial \overline{v^2}}{\partial x} - \frac{\partial}{\partial x}\int_r^\infty \frac{\overline{v^2}-\overline{w^2}}{r}\, dr \tag{A32}$$

With equation (A31) the pressure can be eliminated from equation (A28) and the mean momentum equation becomes;

$$U\frac{\partial U}{\partial x} + V\frac{\partial U}{\partial y} = - \frac{\partial}{\partial x}(\overline{u^2}-\overline{v^2}) - \frac{1}{r}\frac{\partial}{\partial r}(r\overline{uv}) - \frac{\partial}{\partial x}\int_r^\infty \frac{\overline{v^2}-\overline{w^2}}{r}\, dr + \nu\nabla^2 U \tag{A33}$$

Once again if we apply the order of magnitude analysis to the terms in equation (A23) and retain up to the second order terms and neglecting viscous terms the momentum equation becomes,

$$U\frac{\partial U}{\partial x} + V\frac{\partial U}{\partial r} = - \frac{\partial}{\partial x}(\overline{u^2}-\overline{v^2}) - \frac{1}{r}\frac{\partial}{\partial r}(r\overline{uv}) - \frac{\partial}{\partial x}\int_r^\infty \frac{\overline{v^2}-\overline{w^2}}{r}\, dr \tag{A34}$$

However as it will be seen later that the contribution of the first and last terms on the right side of equation (A33) to the mean momentum is insignificant and for most practical problems the momentum equation reduces to

$$U\frac{\partial U}{\partial x} + V\frac{\partial U}{\partial r} = - \frac{1}{r}\frac{\partial}{\partial r}(\overline{uv}) \tag{A35}$$

The Momentum Integral Constraint

Multiplying equation (A33) by r and integrate across the jet (from 0 to ∞) making use of the continuity equation (A26) we have,

$$\int_0^\infty \frac{\partial}{\partial x}[U^2 + \overline{u^2} - \overline{v^2}]\, rdr + \int_0^\infty \frac{\partial}{\partial r}(UV)dr + \int_0^\infty \frac{\partial}{\partial x}\int_r^\infty \frac{\overline{v^2}-\overline{w^2}}{r}\, dr\, rdr$$

(i) (ii) (iii)

$$+ \int_0^\infty \frac{\partial}{\partial r}(r\overline{uv})dr = 0 \qquad \text{(A36)}$$

(iv)

The terms (ii) and (iv) integrate to zero and term (iii) can be integrated by parts so that;

$$\text{(iii)} = \frac{r^2}{2}\int_r^\infty \frac{\overline{v^2}-\overline{w^2}}{r}\Bigg|_0^\infty - \int_0^\infty \frac{r^2}{2}\left(\frac{\overline{v^2}-\overline{w^2}}{r}\right) dr$$

$$= - \int_0^\infty \frac{\overline{v^2}-\overline{w^2}}{2}\, rdr \qquad \text{(A37)}$$

Then the momentum integral becomes;

$$\frac{d}{dx}\int_0^\infty [U^2 + \overline{u^2} - \frac{v^2-w^2}{2}]\, rdr = 0 \qquad \text{(A38)}$$

or at any cross section downstream we must have,

$$2\pi \int_0^\infty [U^2 + \overline{u^2} - \frac{v^2-w^2}{2}]\, rdr = M_o/\rho \qquad \text{(A39)}$$

as a requirement for conservation of momentum where

$$M_o = \rho\pi \frac{U_o^2 d_o^2}{4} \qquad \text{(A40)}$$

U_o is the jet exist velocity and d_o is the jet exit diameter.

APPENDIX B

Analytical Solutions

An analytical solution for the non-linear set of ordinary differential equations that result from the similarity formulation in the applications of the (k-ε)- and stress-models (Chapter 3 and 4) is not feasible at the present time.

However in the similarity region the eddy viscosity ($\nu_t \sim k^2/\varepsilon$) is constant across the flow except at the edge of the shear layer (Tennekes and Lumley (1972)).

The calculated ratio (k^2/ε) is in support of the above statement as it can be seen from figures (3.8)and (3.14). Based on this fact and if we neglect the second order terms in the momentum equation (3.35) we can write;

$$a_1\left(\frac{i+1}{2}\right)\left[f^2 + \frac{1}{\eta_i} f' \int_0^\eta \eta^i f \, d\eta\right] + \frac{1}{\eta^i}\left[(C_\mu K^2/_E)\eta^i f'\right]' = 0 \qquad \text{(B1)}$$

where the eddy viscosity hypothesis is given by,

$$g_4 = -(C_\mu K^2/_E) f' \qquad \text{(B2)}$$

and the momentum integral constraint can be expressed as follows:

$$\int_0^\infty f^2 \eta^i d\eta = \frac{U_o^2}{2U_m^2 \, \ell^{i+1}} \left(\frac{d}{i+1}\right)^{i+1} \qquad \text{(B3)}$$

The parameter a_1 (the similarity constant) can be eliminated if we let $\xi = \sqrt{a_1}\eta$.

In the above notation i=o correspond to the plane jet and i=1 for the axisymmetric case. Now by taking $C_\mu \; K^2/_E$ as constant equation (B1) can be integrated directly and a close form solution can be obtained for both plane and axisymmetric turbulent jet.

I. The Plane Jet

In this case (i=o) the momentum equations becomes

$$\frac{1}{2}\left(f^2+f\int_0^{\xi} f d\xi\right)+\left[(C_\mu K^2/_E)f'\right]' = 0 \tag{B4}$$

and the momentum integral is;

$$\int_0^{\infty} f^2 d\xi = \frac{U_o^2}{2U_m^2}\left(\frac{d_o a_1}{\ell}\right) \tag{B5a}$$

where

$$\ell = a_1 x \tag{B5b}$$

Now if we assume that the centerline velocity is governed by the following decay law:

$$U_m = C\, U_o\left(\frac{d_o}{x}\right)^{1/2} \tag{B6}$$

where C is an empirical constant. Then the momentum integral becomes;

$$\int_0^{\infty} f^2 d\xi = \frac{1}{2C^2} \tag{B7}$$

Let us define

$$G(\xi) = \int_0^{\xi} f(\xi) d\xi \tag{B8}$$

or

$$G'(\xi) = f(\xi) \tag{B9}$$

with the following boundary conditions;

$$G(0) = 0 \tag{B10}$$

$$G'(0) = 1.0 \tag{B11}$$

Integrating the momentum equation once leads to;

$$\frac{1}{2}\, Gf + (C_\mu K^2/E)f' = 0 \tag{B12}$$

where the constant of integration is zero in this case. Based on the assumption that K^2/E remains constant across the jet and making use of equation (B9) we can write equation (B12) as,

$$\frac{1}{2C_\mu K^2/E} GG' + G'' = 0 \tag{B13}$$

By integrating equation (B13) twice and making use of the condition (B10) and (B11) we obtain;

$$G(\xi) = \alpha \tanh(\xi/\alpha) \tag{B14}$$

where

$$\alpha = 2(C_\mu K^2/E)^{1/2} \tag{B15}$$

finally, the velocity profile is given by;

$$f(\xi) = \frac{1}{\cosh^2(\xi/\alpha)} \tag{B16}$$

Now in order to determine α we make use of the momentum integral constraints. Substituting (B16) into (B7) and integrating the resulting equation leads to

$$\int_0^\infty \frac{d\xi}{\cosh^4(\xi/\alpha)} = \frac{\alpha}{4-1} \frac{\sinh(\xi/\alpha)}{\cosh^3(\xi/\alpha)} \Bigg|_0^\infty + \frac{2}{3}\alpha \frac{\sinh(\xi/\alpha)}{\cosh(\xi/\alpha)} \Bigg|_0^\infty = \frac{1}{2C^2} \tag{B17}$$

or

$$\alpha = \frac{3}{4}\frac{1}{C^2}$$

Where the constant C is defined by equation (B6). The value of C can be obtained from experimental data. For example, Gutmark and Wygnanski (1976) data suggests that $C \cong 2.306$. Hence the solution for the plane jet is given by;

$$\frac{U}{U_m} = \frac{1}{\cosh^2(\xi/\alpha)} \tag{B18}$$

$$\frac{\overline{uv}}{U_m^2} = \frac{\alpha}{4.0}\,\frac{\sinh(\xi/\alpha)}{\cosh^3(\xi/\alpha)} \tag{B19}$$

$$\frac{V}{U_m} = \frac{1}{4\cosh^2(\xi/\alpha)}\,[4\xi - \alpha\sinh(2\xi/\alpha)] \tag{B20}$$

II. <u>The Axisymmetric Jet (i=1)</u>

For the round jet the momentum equation becomes;

$$[(C_\mu K^2/E)\xi f']' + \xi G f' + \xi f^2 = 0 \tag{B21}$$

where

$$G = \int_0^\xi f\xi d\xi$$

and the momentum integral is given by;

$$\int_0^\infty f^2\,\xi d\xi = \frac{1}{8}\left(\frac{U_o d_o a_1}{U_m \ell}\right)^2 \tag{B22}$$

if the centerline mean velocity decays like;

$$U_m = C\,\frac{U_o d_o}{x} \tag{B23}$$

where C is an empirical constant and ℓ is given by;

$$\ell = a_1 x \tag{B24}$$

then equation (B22) becomes

$$\int_0^\infty f^2 \xi d\xi = \frac{1}{8C^2} \tag{B25}$$

Integrating equation (B21) once gives

$$fG + (C_\mu K^2/E)\xi f' = 0 \tag{B26}$$

where the constant of integration is zero.

Similarity as in the two-dimensional case if we let $C_\mu K^2/E = \nu_t$ be a constant across the jet, then equation (B26) is satisfied if;

$$f(\xi) = \frac{1}{(1+\frac{1}{8\nu_t}\xi^2)^2} \tag{B27}$$

If we impose the momentum integral constraint on the solution (B27) we can evaluate the width parameter $(\frac{1}{8\nu_t})$ in terms of the empirical constant C. That is;

$$\nu_t = \frac{3}{32C^2} \tag{B28}$$

Hence the exact solution for the round jet is given by;

$$\frac{U}{U_m} = \frac{1}{(1+\frac{4}{3}C^2\xi^2)^2} \tag{B29}$$

$$\frac{\overline{uv}}{U_m^2} = \frac{\xi}{2(+\frac{4C^2}{3}\xi^2)^3} \tag{B30}$$

$$\frac{V}{U_m} = \frac{\xi}{2(1+\frac{4}{3}C^2\xi^2)^2}[1 - \frac{4}{3}C^2\xi^2] \tag{B31}$$

where C is an empirical constant (see Table 3.2)

APPENDIX C

Initial Profiles

The eddy viscosity solution (Appendix B) predicts the velocity and, in turn, the shear stress profile for self-preserving turbulent jets quite well across most of the flow except near the outer edge where the profiles are a little overestimated. However, these profiles will provide an excellent first guess to start the similarity solution in Chapter 3 for the k-ϵ model and Reynolds stress solution in Chapter 4.

The velocity and shear stress are given by the exact expressions (B18),(B19), (B29) and (B30) and they are reviewed in Table (C1). The empirical constant C associated with these profiles for the plane and round jet can be taken from the experimental data. (See Table 3.3.)

For the present calculation an average value of C is given in Table (C2) along with some of the flow characteristics for the plane and the round jet.

Kinetic Energy Estimate

It has been observed from the experimental data of the plane and round jets (i.e. experiments listed in Table 3.2) that at some distance outward from the symmetry axis of the jet until the outer edge of the flow we have the following balance in the energy budget.

$$\text{Production} \cong \text{dissipation}$$

$$\text{Convection transport} \cong \text{diffusion transport}$$

A convenient measure of this distance cited above is the value of ξ where the shear stress is a maximum which is defined here as ξ_{ss} (See Figure C1 and C2). Hence for $\xi \geq \xi_{ss}$ by the first equality given above we have

$$E = - g_4 f' \qquad \text{(C1)}$$

where E is the dissipation rate of energy and g_4 is the shear stress.

By the eddy viscosity hypotheses it follows that the kinetic energy of turbulence is given by

$$K = \left(\frac{C\, g_4}{f'} E\right)^{1/2} \quad ; \quad \xi \geq \xi_{ss} \qquad \text{(C2)}$$

Figure C1 shows the eddy viscosity solution for C = 2.4 and the estimated kinetic energy profile. For $\xi \geq \xi_{ss}$ (the dotted line) it is assumed that the k-profiles decrease as $\xi \to o$ which is based on the experimental data. On the other hand Figure (C2) shows the same results as above but for the round jet and the k-profile is observed to be increasing as $\xi \to o$ based on the round jet data.

Profile	Plane Jet	Round Jet
f	$\dfrac{1}{\cosh^2(\frac{4}{3}C^2\xi)}$	$\dfrac{1}{(1+\frac{4}{3}C^2\xi^2)^2}$
g_4	$\dfrac{3}{16C^2}\tanh(\frac{4}{3}C^2\xi)/f$	$\dfrac{\xi}{2(1+\frac{4}{3}C^2\xi^2)^3}$
f'	$-\dfrac{8C^2}{3}\tanh(\frac{4}{3}C^2\xi)/f$	$-\dfrac{16}{3}C^2\dfrac{\xi}{(1+\frac{4}{3}C^2\xi^2)^3}$
$C_\mu\dfrac{K^2}{E}$	$\dfrac{9}{64C^4}$	$\dfrac{3}{32C^2}$

Table C1. Summary of the Eddy Viscosity Solution.

Flow Constant	Plane Jet	Round Jet
C	2.4	6.0
$(\overline{uv}/U_m)_{max}$	.025	.0186
$\xi_{1/2}$	.115	.093
ξ_{ss}	.085	.065
$\nu_t = .09\ K^2/E$	.00423	.0026

Table C2. Flow Characteristic For the Eddy Viscosity Solution.

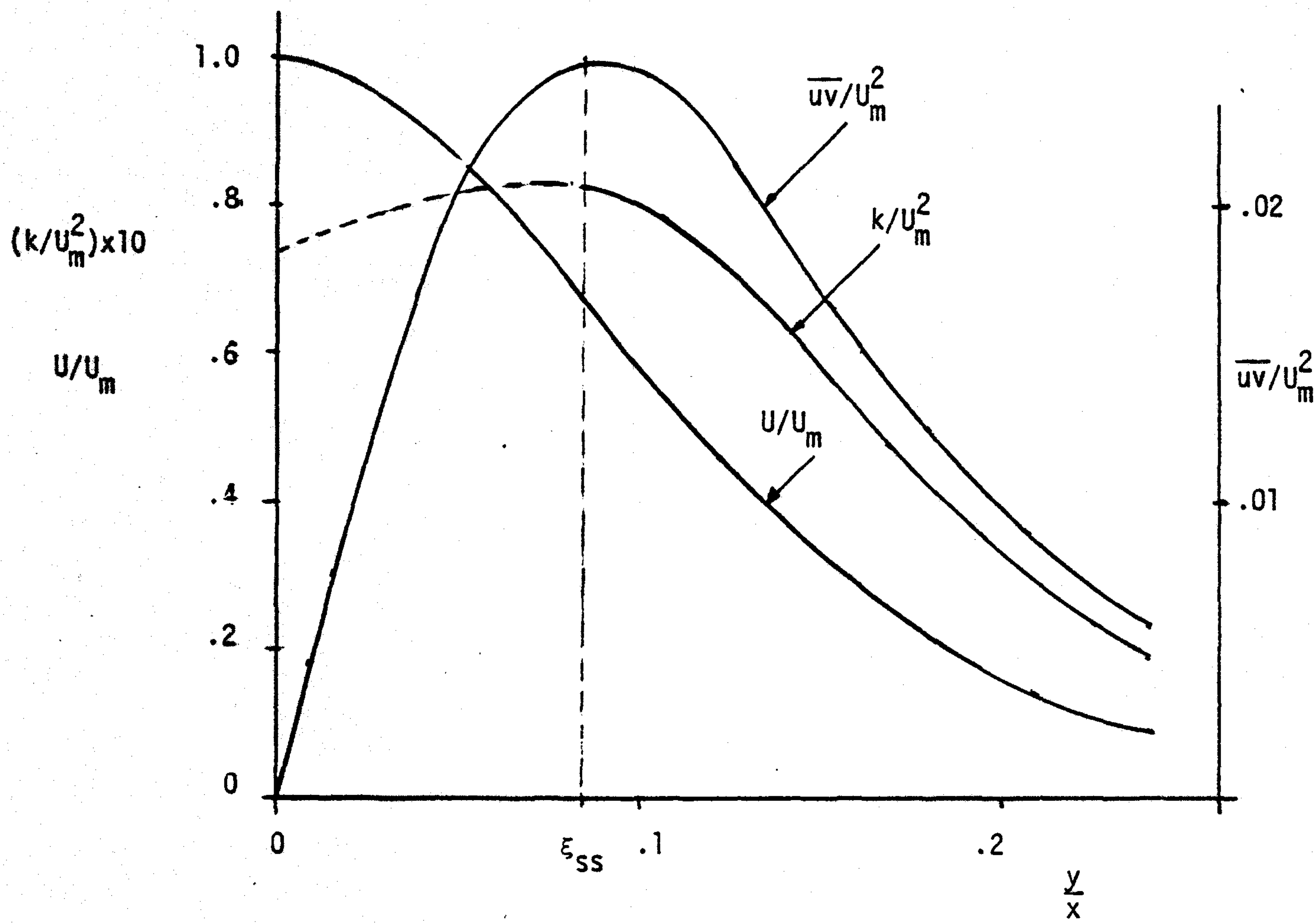

Figure C1. Eddy Viscosity Solution for Self-preserving Two-dimensional Jet.

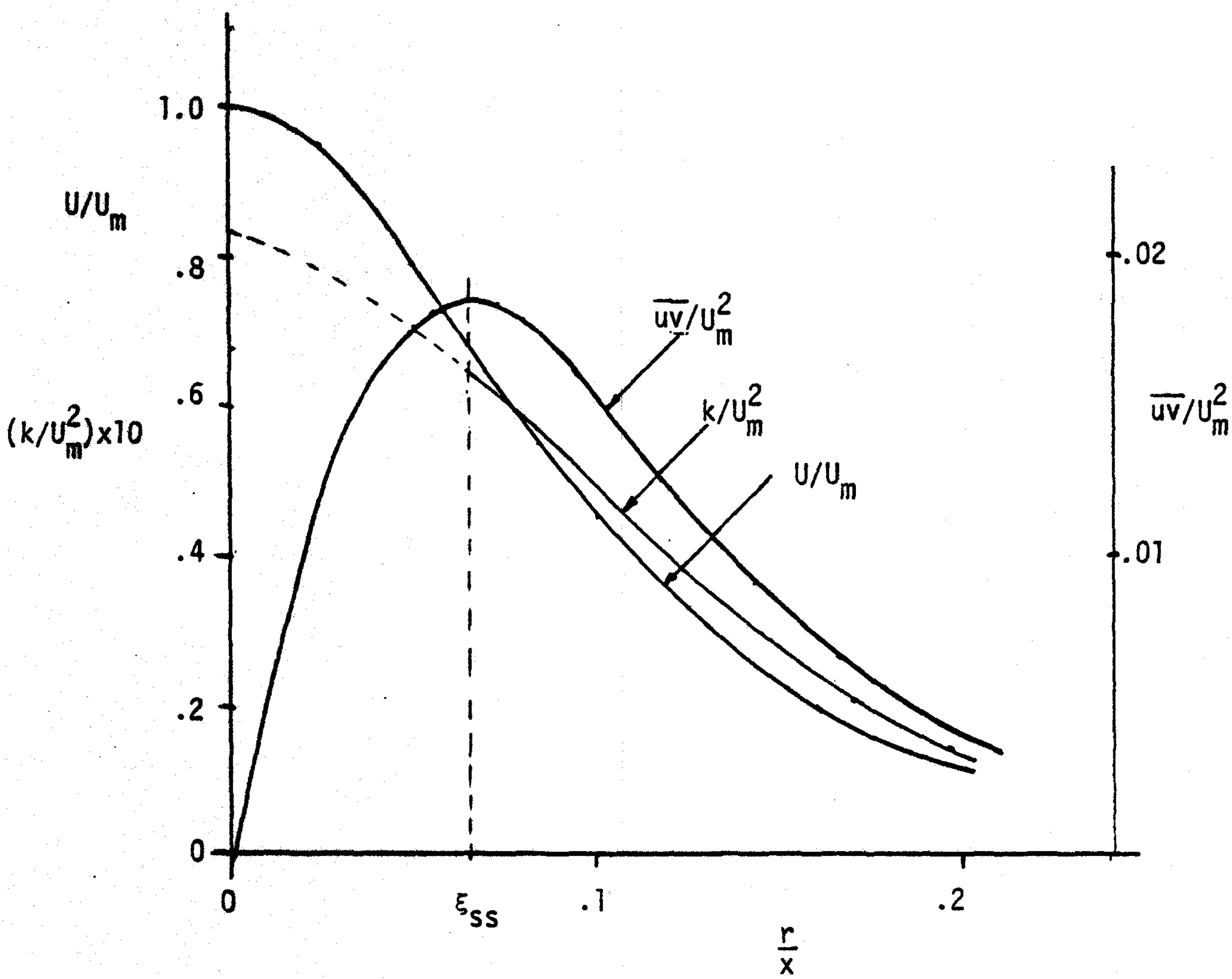

Figure C2. Eddy Viscosity Solution for Self-preserving Axisymmetric Jet.

APPENDIX D

Reynolds Stress Equations in Cartesian Coordinate Systems

The equations for the Reynolds stress components for a general turbulent free shear flow can be obtained from equation (2.67). For steady state motion equation (2.67) can be written as follows;

$$U_k \frac{\partial \overline{u_i u_j}}{\partial x_k} = P_{ij} + \frac{\partial}{\partial x_k} J_{ijk} - C_1 \varepsilon b_{ij} + \phi_{ij} - \frac{2}{3}\varepsilon \delta_{ij} \qquad \text{(D1)}$$

(i) (ii) (iii) (iv) (v) (vi)

The convective term (i), production (ii), return to isotropy (iv) and dissipation (vi) are straightforward and can be easily written in component form. The diffusion transport (iii) can be evaluated from the relations (2.69) and (2.70). The rapid term (v) can be obtained from the fourth order tensor which is given by equation (2.71). Now let us consider a two dimensional flow in which x_1, x_2, x_3 correspond to the cartesian coordinates x, y, and z (Figure A1) and the respective velocity components U_1, U_2, U_3 and u_1,u_2,u_3 correspond to U, V, W, and u.v.w. We assume that there is no mean motion in the z-direction and all derivatives with respect to z are zero. Further the shear stresses $\overline{uw} = \overline{wu}$ and $\overline{vw} = \overline{wv}$ are zero.

Based on the above assumption the dynamic equations for Reynolds stress components become

x-component:

$$U\frac{\partial \overline{u^2}}{\partial x} + V\frac{\partial \overline{u^2}}{\partial y} = -2(\overline{u^2}\frac{\partial U}{\partial x} + \overline{uv}\frac{\partial U}{\partial y}) + \frac{\partial}{\partial x} J_{111} + \frac{\partial}{\partial y} J_{112} - C_1 \varepsilon(\frac{\overline{u^2}}{q^2} - \frac{1}{3})$$

$$+ 4\phi_{11} - \frac{2\varepsilon}{3} \qquad \text{(D2)}$$

y-component:

$$U\frac{\partial \overline{v^2}}{\partial x} + V\frac{\partial \overline{v^2}}{\partial y} = -2(\overline{uv}\ \frac{\partial V}{\partial x} + \overline{v^2}\ \frac{\partial V}{\partial y}) = \frac{\partial}{\partial x} J_{221} + \frac{\partial}{\partial y} J_{222} - C_1\ \varepsilon(\frac{\overline{v^2}}{q^2} - \frac{1}{3})$$

$$+ 4\phi_{22} - \frac{2\varepsilon}{3} \qquad \text{(D3)}$$

z-component:

$$U\frac{\partial \overline{w^2}}{\partial x} + V\frac{\partial \overline{w^2}}{\partial y} = \frac{\partial}{\partial x} J_{331} + \frac{\partial}{\partial y} J_{332} - C_1\ \varepsilon(\frac{\overline{w^2}}{q^2} - \frac{1}{3}) + 4\phi_{33} - \frac{2\varepsilon}{3} \qquad \text{(D4)}$$

Shear Stress:

$$U\frac{\partial \overline{uv}}{\partial x} + V\frac{\partial \overline{uv}}{\partial y} = -\ (\overline{u^2}\ \frac{\partial V}{\partial x} + uv\frac{\partial U}{\partial x} + \overline{v^2}\ \frac{\partial U}{\partial y}) + \frac{\partial}{\partial x} J_{121} + \frac{\partial}{\partial y} J_{132}$$

$$- C_1\varepsilon\ \frac{\overline{uv}}{q^2} + \phi_{12} \qquad \text{(D5)}$$

The transport and rapid terms in the above system of equations involve a lot of terms. However, some of them are negligibly small compared to the leading terms and they will be dropped out. Hence before we evaluate these terms let us take a look at the order of magnitude of the various terms in the above equations. However, since the terms in the proceeding equations are similar it will be sufficient to examine equation (D5) and the same analysis will be applicable to the rest of the equations.

Using the same scales as in Appendix A we may scale equation (D5) as follows,

$$U\frac{\partial \overline{u^2}}{\partial x} + V\frac{\partial \overline{uv}}{\partial y} = -\ \overline{u^2}\ \frac{\partial V}{\partial x} - uv\ \frac{\partial U}{\partial x} - uv\ \frac{\partial V}{\partial y} - \overline{v^2}\ \frac{\partial U}{\partial y}$$

$$\frac{U_m u^2}{\ell}[\quad \frac{\ell}{L} \qquad \frac{\ell}{L} \quad = \quad (\frac{\ell}{L})^2 \qquad \frac{\ell}{L} \qquad \frac{\ell}{L} \qquad 1$$

$$+ \frac{\partial J_{121}}{\partial x} + \frac{\partial J_{122}}{\partial y} - C_1 \, \varepsilon \frac{\overline{uv}}{q^2} + \phi_{12}$$

$$\frac{u}{U_m}\frac{\ell}{L} \qquad \frac{u}{U_m} \qquad \frac{u}{U_m} \qquad 1] \tag{D6}$$

In the above scaling the diffusive transport and rapid term has been represented by their largest term. From previous analysis we have,

$$(\frac{u}{U_m})^2 \sim \frac{\ell}{L} << 1$$

Hence the first term in the production is negligibly small so is the first diffusion term. The rest of the terms are small compared to unity but they are of the same order as the left hand side of the equation. Hence, we are keeping these terms for later analysis in Chapter 4 and 5. Based on the above analysis and the assumption made earlier we will evaluate the rapid and transport terms.

Rapid Terms

The mean velocity gradients for the two-dimensional flow is given by

$$U_{p,q} = \begin{vmatrix} \frac{\partial U}{\partial x} & \frac{\partial U}{\partial y} & 0 \\ 0 & \frac{\partial V}{\partial y} & 0 \\ 0 & 0 & 0 \end{vmatrix} \tag{D7}$$

Hence, from equation (2.71) and (2.72) the rapid terms can be evaluated. Based on our assumption above and with equation (D7) these terms become

$$\phi_{11} = [\frac{2}{30} - c(\frac{\overline{u^2}}{\overline{q^2}} - \frac{1}{3})] \, \overline{q^2} \, \frac{\partial U}{\partial x} + \frac{1}{3}(1+2c) \, uv \, \frac{\partial U}{\partial y} + [- \frac{1}{30} + c(\frac{\overline{u^2}}{\overline{q^2}} - \frac{1}{3}) + c(\frac{\overline{v^2}}{\overline{q^2}} - \frac{1}{3})] \, \overline{q^2} \, \frac{\partial V}{\partial y} \tag{D8a}$$

$$\phi_{22} = [- \frac{1}{30} + c(\frac{\overline{u^2}}{\overline{q^2}} - \frac{1}{3}) + c(\frac{\overline{v^2}}{\overline{q^2}} - \frac{1}{3})] \overline{q^2} \frac{\partial U}{\partial x} - \frac{1}{3}(1+5c) \overline{uv} \frac{\partial U}{\partial y}$$

$$+ [\frac{1}{30} + c(\frac{\overline{u^2}}{\overline{q^2}} - \frac{1}{3})] \overline{q^2} \frac{\partial V}{\partial y} \quad \text{(D8b)}$$

$$\phi_{33} = [- \frac{1}{30} + c(\frac{\overline{u^2}}{\overline{q^2}} - \frac{1}{3}) + c(\frac{\overline{w^2}}{\overline{q^2}} + \frac{1}{3})] \overline{q^2} \frac{\partial U}{\partial x} + c\ uv \frac{\partial U}{\partial y}$$

$$+ [\frac{1}{30} + c(\frac{\overline{v^2}}{\overline{q^2}} - \frac{1}{3}) + c(\frac{\overline{w^2}}{\overline{q^2}} - \frac{1}{3})] \overline{q^2} \frac{\partial V}{\partial y} \quad \text{(D8c)}$$

$$\phi_{12} = [\frac{1}{10} + (1-c) \frac{\overline{v^2}}{\overline{q^2}} - \frac{1}{3} (1+8c) \frac{\overline{u^2}}{\overline{q^2}} + c] \overline{q^2} \frac{\partial U}{\partial y} \quad \text{(D8d)}$$

<u>Transport Terms</u>

The transport terms are given by (see Section 2.12)

$$J_{ijk} = \frac{2}{3(2-b)C_1} \frac{\overline{q^2}}{\varepsilon} G_{ijk} - \frac{1}{(2-9b/4)C_1+5} \frac{\overline{q^2}}{\varepsilon} \quad \frac{C_1-2}{3(2-b)C_1} G_k \delta_{ij}$$

$$+ (\frac{C_1-2}{3(2-b)C_1})(G_j\delta_{ik} + G_i\delta_{jk})] \quad \text{(D9)}$$

where G_{ijk} and G_i are given by equation (2.67) and (2.70). The non-zero terms in the functions G_{ijk} and G_i are

$$G_1 = uv \frac{\partial \overline{q^2}}{\partial y} + 2uv \frac{\partial \overline{u^2}}{\partial y} + 2\ \overline{v^2} \frac{\partial \overline{uv}}{\partial y} \quad \text{(D10a)}$$

$$G_2 = \overline{v^2} \frac{\partial \overline{q^2}}{\partial y} + 2\overline{uv} \frac{\partial \overline{uv}}{\partial y} + 2\ \overline{v^2} \frac{\partial \overline{v^2}}{\partial y} \quad \text{(D10b)}$$

$$G_{112} = \overline{v^2} \frac{\partial \overline{u^2}}{\partial y} + 2\ uv \frac{\partial uv}{\partial y} \quad \text{(D10c)}$$

$$G_{222} = 3\ \overline{v^2} \frac{\partial \overline{v^2}}{\partial y} \quad \text{(D10e)}$$

$$G_{332} = \overline{v^2} \frac{\partial \overline{w^2}}{\partial y} \quad \text{(D10f)}$$

$$G_{122} = \overline{v^2}\,\frac{\partial uv}{\partial y} + \overline{v^2}\,\frac{\partial \overline{uv}}{\partial y} + \overline{uv}\,\frac{\partial \overline{v^2}}{\partial y} \tag{D10g}$$

If we substitute (D10) into (D9) the diffusion terms become

$$J_{112} = C_o\,\frac{\overline{q^2}}{\varepsilon}\,[(C_2+1)\,\overline{v^2}\,\frac{\partial \overline{u^2}}{\partial y} + 3C_2\,\overline{v^2}\,\frac{\partial \overline{v^2}}{\partial y} + C_2\,\overline{v^2}\,\frac{\partial \overline{w^2}}{\partial y} + 2(C_2+1)\,\overline{uv}\,\frac{\partial \overline{uv}}{\partial y}\,] \tag{D11a}$$

$$J_{222} = C_o\,\frac{\overline{q^2}}{\varepsilon}\,[3(C_2+\frac{3}{5})\,\overline{v^2}\,\frac{\partial \overline{v^2}}{\partial y} + (C_2-\frac{2}{5})\,\overline{v^2}\,\frac{\partial \overline{u^2}}{\partial y} + (C_2-\frac{2}{5})\,\overline{v^2}\,\frac{\partial \overline{w^2}}{\partial y} + 2(C_2-\frac{2}{5})\,\overline{uv}\,\frac{\partial \overline{uv}}{\partial y} \tag{D11b}$$

$$J_{332} = C_o\,\frac{\overline{q^2}}{\varepsilon}\,[(C_2+1)\,\overline{v^2}\,\frac{\partial \overline{w^2}}{\partial y} + 3C_2\,\overline{v^2}\,\frac{\partial \overline{u^2}}{\partial y} + 2\,\overline{uv}\,\frac{\partial \overline{uv}}{\partial y}] \tag{D11c}$$

$$J_{122} = C_o\,\frac{\overline{q^2}}{\varepsilon}\,[\frac{8}{5}\,\overline{v^2}\,\frac{\partial \overline{uv}}{\partial y} + \frac{4}{5}\,\overline{uv}\,\frac{\partial \overline{v^2}}{\partial y} - \frac{3}{5}\,\overline{uv}\,\frac{\partial \overline{u^2}}{\partial y} - \frac{1}{5}\,\overline{uv}\,\frac{\partial \overline{w^2}}{\partial y} \tag{D11d}$$

where

$$C_o = \frac{1}{3(2-b)C_1} \tag{D12}$$

$$C_2 = \frac{2(C_1-2)}{20+C_1(8-9b)} \tag{D13}$$

Finally the dynamic equations for the Reynolds stress components in two-dimensional turbulent flow become,

$$U \frac{\partial \overline{u^2}}{\partial x} + V \frac{\partial \overline{u^2}}{\partial y} = - 2\overline{u^2} \frac{\partial U}{\partial x} - 2\overline{uv} \frac{\partial U}{\partial y} + \frac{\partial}{\partial y} [C_o \frac{\overline{q^2}}{\varepsilon} \{\overline{v^2}(C_2+1) \frac{\partial \overline{u^2}}{\partial y}$$

$$+ 3C_2 \overline{v^2} \frac{\partial \overline{v^2}}{\partial y} + C_o \overline{v^2} \frac{\partial \overline{w^2}}{\partial y} + 2(C_2+1) \overline{uv} \frac{\partial \overline{uv}}{\partial y}\}]$$

$$+ 4[\frac{2}{30} - c(\frac{\overline{u^2}}{q^2} - \frac{1}{3})] \overline{q^2} \frac{\partial U}{\partial x} + \frac{4}{3}(1+26) \overline{uv} \frac{\partial U}{\partial y}$$

$$+ 4[- \frac{1}{30} + c(\frac{\overline{u^2}}{q^2} - \frac{1}{3}) + c(\frac{\overline{v^2}}{q^2} - \frac{1}{3})] \overline{q^2} \frac{\partial V}{\partial y} - C_1\varepsilon(\frac{\overline{u^2}}{q^2} - \frac{1}{3}) - \frac{2\varepsilon}{3}$$

(D14a)

$$U \frac{\partial \overline{v^2}}{\partial x} + V \frac{\partial \overline{v^2}}{\partial y} = - 2\overline{v^2} \frac{\partial V}{\partial y} \frac{\partial}{\partial y} [C_o \frac{\overline{q^2}}{\varepsilon} \{3(C_2+\frac{3}{5})\overline{v^2} \frac{\partial \overline{v^2}}{\partial y} + (C_2-\frac{2}{3})\overline{v^2} \frac{\partial \overline{u^2}}{\partial y}$$

$$+ (C_2-\frac{2}{5})\overline{v^2} \frac{\partial \overline{w^2}}{\partial y} + 2(C_2-\frac{2}{5})\overline{uv} \frac{\partial \overline{uv}}{\partial y}] + 4[- \frac{1}{30} + c(\frac{\overline{u^2}}{q^2} - \frac{1}{3})$$

$$+ c(\frac{\overline{v^2}}{q^2} - \frac{1}{3})]\overline{q^2} \frac{\partial U}{\partial x} - \frac{4}{3} (1+5c)\overline{uv} \frac{\partial U}{\partial y} + 4[- \frac{1}{30}$$

$$+ c(\frac{\overline{u^2}}{q^2} - \frac{1}{3})]\overline{q^2} \frac{\partial V}{\partial y} - C_1\varepsilon(\frac{\overline{v^2}}{q^2} - \frac{1}{3}) - \frac{2\varepsilon}{3} \qquad \text{(D14b)}$$

$$U \frac{\partial \overline{w^2}}{\partial x} + V \frac{\partial \overline{w^2}}{\partial y} = \frac{\partial}{\partial y} [C_o \frac{\overline{q^2}}{\varepsilon} \{(C_2+1)\overline{v^2} \frac{\partial \overline{w^2}}{\partial y} + 3C_2 \overline{v^2} \frac{\partial \overline{v^2}}{\partial y} + C_2 \overline{v^2} \frac{\partial \overline{u^2}}{\partial y}$$

$$+ 2\overline{uv} \frac{\partial \overline{uv}}{\partial y}\}] + 4[- \frac{1}{30} + c(\frac{\overline{u^2}}{q^2} - \frac{1}{3}) + c(\frac{\overline{w^2}}{q^2} - \frac{1}{3})]\overline{q^2} \frac{\partial U}{\partial x}$$

$$+ 4c \overline{uv} \frac{\partial U}{\partial y} + 4[- \frac{1}{30} + c(\frac{\overline{v^2}}{q^2} - \frac{1}{3}) + c(\frac{\overline{w^2}}{q^2} - \frac{1}{3})]\overline{q^2} \frac{\partial V}{\partial y}$$

$$- C_1\varepsilon(\frac{\overline{w^2}}{q^2} - \frac{1}{3}) - \frac{2\varepsilon}{3} \qquad \text{(D14c)}$$

$$U \frac{\partial \overline{uv}}{\partial x} + V \frac{\partial \overline{uv}}{\partial y} = - \overline{uv} \frac{\partial U}{\partial x} - \overline{v^2} \frac{\partial U}{\partial y} + \frac{\partial}{\partial y} [C_o \frac{\overline{q^2}}{\varepsilon} \{\frac{8}{5} \overline{u^2} \frac{\partial \overline{uv}}{\partial y}$$

$$+ \frac{4}{5} \overline{uv} \frac{\partial \overline{v^2}}{\partial y} - \frac{3}{5} \overline{uv} \frac{\partial \overline{u^2}}{\partial y} - \frac{1}{5} \overline{uv} \frac{\partial \overline{w^2}}{\partial y}\}]$$

$$+ 2[\frac{1}{10} + (1-c) \frac{\overline{v^2}}{\overline{q^2}} - \frac{1}{3} (1+8c) \frac{\overline{u^2}}{\overline{q^2}} + c]\overline{q^2} \frac{\partial U}{\partial y}$$

$$- C_1 \varepsilon \frac{\overline{uv}}{\overline{q^2}} \tag{D14e}$$

where the underlined terms are of the 2nd order and C_o and C_2 are given by (D12) and (D13).

APPENDIX E

Reynolds Stress Equations in Cylindrical Coordinate System

For the axisymmetric turbulent flow let us write equation (2.67) in curvlinear tensor form. Setting $\frac{\partial}{\partial t} = 0$ the Reynolds stress equation can be transformed to covarient form as follows.

$$U^k \overline{u_i u_j}_{,k} = P^k_{ij} + J^k_{ij,k} + C_1 \; \varepsilon b_{ij} + \phi^{pq}_{ij} - \frac{2}{3} \varepsilon g_{ij} \tag{E1}$$

where

$$P^k_{ij} = -\overline{u^k u_j} \; U_{i,k} - \overline{u^k u_i} \; U_{j,k} \tag{E2}$$

$$J_{ij}{}^k = - C_o \overline{q^2} \; [G^k_{ij} + C_2 \; g^{\ell m} G^k_{\ell m} \; g_{ij} - \frac{1}{5} \; (g^{\ell m} G^j_{\ell m} g_{ik} + g^{\ell m} G^i_{\ell m} g_{jk})] \tag{E3}$$

$$G^k_{ij} = \overline{u^k u^p} (\overline{u_i u_j})_{,p} + \overline{u_j u^p} (\overline{u_i u^k})_{,p} + \overline{u_i u^p} (\overline{u_j u^k})_{,p} \tag{E4}$$

$$\begin{aligned} I^{pp}_{ij} = & - \frac{1}{3} (b^j_i \; \delta^q_j - b^q_j \; \delta^p_i) + \frac{1}{30} \; (4 \delta^p_i \; \delta^q_j - \delta^p_s \; g^{sq} g_{ij} - \delta^p_j \; \delta^q_i) \\ & + c[b^{pq} g_{ji} + b^q_i \delta^p_j + b^p_j \delta^q_i + \delta^p_s g^{sq} b_{ij} - \frac{11}{3} \; b^p_i \delta^q_j \\ & - \frac{4}{3} \; b^q_j \delta^p_i] \end{aligned} \tag{E5}$$

$$\phi^{pq}_{ij} = 2(I^{pq}_{ij} + I^{pq}_{ji}) \; \overline{q^2} \; U_{p,q} \tag{E6}$$

and δ^j_i, g_{ij} and g^{ij} can be obtained by the following transformation relations.

Let (x^1, x^2, x^3) correspond to the cylindrical coordinates (r, θ, z) and the respective mean and turbulent velocity components are (U_1, U_2, U_3) and (u_1, u_2, u_3) which correspond to (U, V, W) and (u, v, w), ξ^1, ξ^2 and ξ^3 correspond to the cartesian coordinate (x, y, z) and the respective velocity components are (U(1), U(2), U(3)) and (u(1), u(2), u(3)). Then we have the following transformation relations.

$$x^1 = [(\xi^1)^2 + (\xi^2)^2]^{1/2}$$

$$x^2 = \tan^{-1} \xi^2/\xi^1$$

$$x^3 = \xi^3$$

From tensor algebra we have,

$$g_{ij} = \frac{\partial \xi^k}{\partial x^i} \frac{\partial \xi^k}{\partial x^j} \quad ; \quad g_{ij} = o, \; i \neq j$$

$$\Gamma^i_{ij} = \frac{1}{h_i} \frac{\partial h_i}{\partial x^j} \quad ; \quad \Gamma^k_{ij} = 0, \; i \neq j \neq k$$

$$\Gamma^k_{jj} = \frac{h_j}{h_k} \frac{\partial h_j}{\partial x^k}$$

$$\Gamma^j_{jj} = \frac{1}{h_j} \frac{\partial h_j}{\partial x^j} \tag{E7}$$

$$g_{ij} g^{ij} = 1$$

$$\delta^j_i = \begin{matrix} 0 & i \neq j \\ 1 & i = j \end{matrix}$$

$$g_{11} = h_1^2 \; ; \; g_{22} = h_2^2 \; ; \; g_{33}^2 = h_3^2$$

$$h_1 = 1 \; , \; h_2 = x^1 \; , \; h_3 = 1$$

$$u = h_1 \, u(1) \; , \; v = h_2 \, u(2) \; , \; w = h_3 \, u(3)$$

$$u^1 = \frac{u}{h_1} \; , \; u^2 = \frac{v}{h_2} \; , \; u^3 = \frac{w}{h_3}$$

Now let us evaluate equation (E4) while observing the tensors differentiation. Hence,

$$G^k_{ij} = \overline{u^k u^p}\,[\frac{\partial \overline{u_i u_j}}{\partial x p} - \Gamma^m_{ip}\,\overline{u_m u_j} - \Gamma^m_{ip}\,\overline{u_i u_m}]$$

$$+ \overline{u_j u^p}\,[\frac{\partial \overline{u_i u^k}}{\partial x p} + \Gamma^k_{mp}\,\overline{u^m u_i} - \Gamma^m_{ip}\,\overline{u^k u_m}]$$

$$+ \overline{u_i u^p}\,[\frac{\partial \overline{u_i u^k}}{\partial x p} + \Gamma^k_{mp}\,\overline{u^m u_j} - \Gamma^m_{jp}\,\overline{u^k u_m}] \qquad \text{(E8)}$$

Or in component form we have

$$G^1_{11} = 3\,\overline{u^2}\,\frac{\partial \overline{u^2}}{\partial r} + \frac{\overline{uv}}{r}\,\frac{\partial \overline{u^2}}{\partial \theta} + 3\,\overline{uw}\,\frac{\partial \overline{u^2}}{\partial z} - 6\,\frac{\overline{uv}^2}{r} \qquad \text{(E9a)}$$

$$G^1_{22} = \overline{u^2}\,\frac{\partial r^2\overline{v^2}}{\partial r} + \frac{\overline{uv}}{r}\,\frac{\partial}{\partial \theta}\,r^2\overline{v^2} + \overline{uw}\,\frac{\partial r^2\overline{v^2}}{\partial z} + 2r\,\overline{uv}\,\frac{\partial r\overline{uv}}{\partial r}$$

$$+ 2\,\overline{v^2}\,\frac{\partial r\overline{uv}}{\partial \theta} + 2\,r\overline{vw}\,\frac{\partial r\overline{uv}}{\partial z} - 2r\,\overline{v^2} - 2r\,\overline{uv}^2 \qquad \text{(E9b)}$$

$$G^1_{33} = \overline{u^2}\,\frac{\partial \overline{w^2}}{\partial r} + \frac{\overline{uv}}{r}\,\frac{\partial \overline{w^2}}{\partial \theta} + \overline{uw}\,\frac{\partial \overline{w^2}}{\partial z} + 2\,\overline{uw}\,\frac{\partial \overline{wu}}{\partial r} + 2\,\frac{\overline{vw}}{r}\,\frac{\partial \overline{wu}}{\partial \theta}$$

$$+ 2\,\overline{w^2}\,\frac{\partial \overline{uw}}{\partial z} - \frac{2}{r}\,\overline{vw^2} \qquad \text{(E9c)}$$

$$G^1_{13} = \overline{u^2}\,\frac{\partial \overline{uw}}{\partial r} + \frac{\overline{uv}}{r}\,\frac{\partial \overline{uw}}{\partial \theta} + \overline{uw}\,\frac{\partial \overline{uw}}{\partial z} + \overline{uw}\,\frac{\partial \overline{u^2}}{\partial r} + \frac{\overline{vw}}{r}\,\frac{\partial \overline{u^2}}{\partial \theta} + \overline{w^2}\,\frac{\partial \overline{u^2}}{\partial z}$$

$$+ \overline{u^2}\,\frac{\partial \overline{uw}}{\partial r} + \frac{\overline{uv}}{r}\,\frac{\partial \overline{uw}}{\partial \theta} + \overline{uw}\,\frac{\partial \overline{uw}}{\partial z} - \frac{4}{r}\,\overline{uv}\,\overline{vw} \qquad \text{(E9d)}$$

$$G^2_{12} = \frac{1}{r}\,\overline{vu}\,\frac{\partial r\overline{uv}}{\partial r} + \frac{\overline{v^2}}{r}\,\frac{\partial r\overline{uv}}{\partial \theta} + \frac{1}{r}\,\overline{vw}\,\frac{\partial r\overline{uv}}{\partial z}$$

$$+ r\overline{vu}\,\frac{\partial}{\partial r}\,\frac{\overline{uv}}{r} + \overline{v^2}\,\frac{\partial}{\partial \theta}\,\frac{\overline{uv}}{r} + r\overline{vw}\,\frac{\partial}{\partial z}\,\frac{\overline{uv}}{r} + \overline{u^2}\,\frac{\partial \overline{v^2}}{\partial r}$$

$$+ \frac{\overline{uv}}{r}\,\frac{\partial \overline{v^2}}{\partial \theta} + \overline{uw}\,\frac{\partial \overline{v^2}}{\partial z} + \frac{2}{r}\,\overline{u^2}\,\overline{v^2} - \frac{2}{r}\,\overline{v^2}^2 + \frac{2}{r}\,\overline{uv}^2 \qquad \text{(E9e)}$$

$$G^3_{11} = \overline{uw}\,\frac{\partial\overline{u^2}}{\partial r} + \frac{1}{r}\,\overline{vw}\,\frac{\partial\overline{u^2}}{\partial\theta} + \overline{w^2}\,\frac{\partial\overline{u^2}}{\partial z} + 2\overline{u^2}\,\frac{\partial\overline{uw}}{\partial r} + 2\,\frac{1}{r}\,\overline{uv}\,\frac{\partial\overline{uw}}{\partial\theta}$$

$$+ 2\overline{uw}\,\frac{\partial\overline{uw}}{\partial z} - \frac{4}{r}\,\overline{uv}\;\overline{vw} \qquad \text{(E9f)}$$

$$G^3_{22} = \overline{uw}\,\frac{\partial r^2\overline{v^2}}{\partial r} + \frac{\overline{vw}}{r}\,\frac{\partial r^2\overline{v^2}}{\partial\theta} + \overline{w^2}\,\frac{\partial r^2\overline{v^2}}{\partial z} + 2r\,\overline{uv}\,\frac{\partial r\overline{vw}}{\partial r}$$

$$+ 2\overline{v^2}\,\frac{\partial r\overline{vw}}{\partial\theta} + 2r\,\overline{vw}\,\frac{\partial r\overline{vw}}{\partial z} \qquad \text{(E9g)}$$

$$G^3_{33} = 3\overline{uw}\,\frac{\partial\overline{w^2}}{\partial r} + 3\,\frac{\overline{vw}}{r}\,\frac{\partial}{\partial\theta}\,\overline{w^2} + 3\,\overline{w^2}\,\frac{\partial\overline{w^2}}{\partial z} \qquad \text{(E9h)}$$

$$G^2_{23} = \frac{1}{r}\,\overline{uv}\,\frac{\partial}{\partial r}\,r\overline{vw} + \frac{1}{r^2}\,\overline{v^2}\,\frac{\partial}{\partial\theta}\,r\overline{vw} + \frac{1}{r}\,\overline{vw}\,\frac{\partial}{\partial z}\,\overline{vw}$$

$$+ \overline{uv}\,\frac{\partial\overline{v^2}}{\partial r} + \frac{\overline{vw}}{r}\,\frac{\partial\overline{v^2}}{\partial\theta} + \overline{w^2}\,\frac{\partial\overline{v^2}}{\partial z} + r\overline{uv}\,\frac{\partial}{\partial r}\,\frac{\overline{vw}}{r}$$

$$+ \overline{v^2}\,\frac{\partial}{\partial\theta}\,\frac{\overline{vw}}{r} + r\overline{vw}\,\frac{\partial}{\partial z}\,\frac{vw}{r} + \frac{2}{r}\,\overline{uv}\;\overline{uw} + \frac{2}{r}\,\overline{v^2}\;\overline{uw} \qquad \text{(E9i)}$$

Consider now only axisymmetric turbulent shear flow without swirl so that all shear stresses but ($\overline{uw} = \overline{wu}$) are zero. Further $\frac{\partial}{\partial\theta} = 0$. The equations in (E9) simplify to;

$$G^1_{11} = 3\,\overline{u^2}\,\frac{\partial\overline{u^2}}{\partial r} \qquad \text{(E10a)}$$

$$G^1_{22} = \overline{u^2}\,\frac{\partial}{\partial r}\,r^2\overline{v^2} - 2r\,\overline{v^2}^2$$

$$= r^2\overline{u^2}\,\frac{\partial\overline{v^2}}{\partial r} + 2r\,(\overline{v^2}\,\overline{u^2} - \overline{v^2}^2) \qquad \text{(E10b)}$$

$$G^1_{33} = \overline{u^2}\,\frac{\partial\overline{w^2}}{\partial r} + 2\overline{uw}\,\frac{\partial\overline{uw}}{\partial r} \qquad \text{(E10c)}$$

$$G^1_{13} = 2\overline{u^2}\,\frac{\partial\overline{uw}}{\partial r} + \overline{uw}\,\frac{\partial\overline{u^2}}{\partial r} \qquad \text{(E10d)}$$

$$G^2_{12} = \overline{u^2}\,\frac{\partial\overline{v^2}}{\partial r} + \frac{2}{r}\,(\overline{u^2}\,\overline{v^2} - \overline{v^2}^2) \qquad \text{(E10e)}$$

$$G^3_{11} = \overline{uw}\,\frac{\partial\overline{u^2}}{\partial r} + 2\overline{u^2}\,\frac{\partial\overline{uw}}{\partial r} \qquad \text{(E10f)}$$

$$G_{22}^{3} = \overline{uw}\,\frac{\partial}{\partial r}\,r^2\overline{v^2} \tag{E10g}$$

$$G_{33}^{3} = 3\overline{uw}\,\frac{\partial\overline{w^2}}{\partial r} \tag{E10h}$$

$$G_{23}^{2} = \overline{uw}\,\frac{\partial\overline{v^2}}{\partial r} + \frac{2}{r}\,\overline{v^2}\;\overline{uw} \tag{E10i}$$

<u>Diffusion Terms</u>

$$J_{ij,k}^{k} = \frac{\partial}{\partial x^k}\,J_{ij}^{k} + \Gamma_{mk}^{k}\,J_{ij}^{m} - \Gamma_{ik}^{m}\,J_{mj}^{k} - \Gamma_{jk}^{m}\,J_{im}^{k} \tag{E11}$$

If we substitute for the Γ's we get

$$J_{11,k}^{k} = \frac{\partial J_{11}^{k}}{\partial x^k} + \frac{1}{r}\,J_{11}^{1} - \frac{2}{r}\,J_{12}^{2} \tag{E12a}$$

$$J_{22,k}^{k} = \frac{\partial J_{22}^{k}}{\partial x^k} + \frac{1}{r}\,J_{22}^{1} - \frac{2}{r}\,J_{22}^{1} + 2r\,J_{12}^{2} \tag{E12b}$$

$$J_{33,k}^{k} = \frac{\partial J_{33}^{k}}{\partial x^k} + \frac{1}{r}\,J_{33}^{1} \tag{E12c}$$

$$J_{13,k}^{k} = \frac{\partial J_{13}^{k}}{\partial x^k} + \frac{1}{r}\,J_{13}^{1} - \frac{1}{r}\,J_{23}^{2} \tag{E12d}$$

From (E3) and (E10) the J's can be evaluated and the results are;

$$J_{11}^{1} = C_o\frac{\overline{q^2}}{\varepsilon}\,[3(C_2+\tfrac{3}{5})\,\overline{u^2}\,\frac{\partial\overline{u^2}}{\partial r} + (C_2-\tfrac{2}{5})\,\{\overline{u^2}\,\frac{\partial\overline{v}}{\partial r} + \frac{2}{r}(\overline{u^2v^2}-\overline{v^2}^2) + \overline{u^2}\,\frac{\partial\overline{w^2}}{\partial r} + 2\,\overline{uw}\,\frac{\partial\overline{uw}}{\partial r}\}] \tag{E13a}$$

$$J_{12}^{2} = C_o\,\frac{\overline{q^2}}{\varepsilon}\,[\overline{u^2}\,\frac{\partial\overline{v^2}}{\partial r}\,(\overline{u^2v^2}-\overline{v^2}^2) - \frac{1}{5}\,\{3\,\overline{u^2}\,\frac{\partial\overline{u^2}}{\partial r} + \overline{u^2}\,\frac{\partial\overline{v^2}}{\partial r} + \frac{2}{r}\,(\overline{u^2v^2}-\overline{v^2}^2) + u^2\,\frac{\partial\overline{w^2}}{\partial r} + 2\overline{uw}\,\frac{\partial\overline{uw}}{\partial r}\}] \tag{E13b}$$

$$J^1_{22} = -C_o \frac{\overline{q^2}}{\varepsilon} [\overline{r^2}\, C_o\, 3\overline{u^2} \frac{\partial \overline{u^2}}{\partial r} + (C_2+1)\, \{r^2\, \overline{u^2} \frac{\partial \overline{v^2}}{\partial r}$$

$$+ 2r\, (\overline{u^2 v^2} - \overline{v^2}^2)\} + r^2\, C_2\, \{\overline{u^2} \frac{\partial \overline{w^2}}{\partial r} + 2\overline{uw} \frac{\partial \overline{uw}}{\partial r}\}] \qquad \text{(E13c)}$$

$$J^1_{33} = -C_o \frac{\overline{q^2}}{\varepsilon} [(C_2+1)(\overline{u^2} \frac{\partial \overline{w^2}}{\partial r} + 2\overline{uw} \frac{\partial \overline{uw}}{\partial r}) + C_2\, 3\overline{u^2} \frac{\partial \overline{u^2}}{\partial r}$$

$$+ C_2\, \{\overline{u^2} \frac{\partial \overline{v^2}}{\partial r} + \frac{2}{r} (\overline{u^2 v^2} - \overline{v^2}^2)\}] \qquad \text{(E13d)}$$

$$J^1_{13} = -C_o \frac{\overline{q^2}}{\varepsilon} [2\overline{u^2} \frac{\partial \overline{uw}}{\partial r} + \overline{uw} \frac{\partial \overline{u^2}}{\partial r} - \frac{1}{5} \{\overline{uw} \frac{\partial \overline{u^2}}{\partial r} + 2\overline{u^2} \frac{\partial \overline{uw}}{\partial r}$$

$$+ \overline{uw} \frac{\partial \overline{v^2}}{\partial r} + \frac{2}{r} \overline{uw}\, \overline{v^2} + 3\overline{uw} \frac{\partial \overline{w^2}}{\partial r}\}] \qquad \text{(E13e)}$$

$$J^2_{23} = -C_o \frac{\overline{q^2}}{\varepsilon} [\overline{uw} \frac{\partial \overline{v^2}}{\partial r} + \frac{2}{r} \overline{v^2}\, \overline{uw} - \frac{1}{5} \{\overline{uw} \frac{\partial \overline{u^2}}{\partial r} + 2\overline{u^2} \frac{\partial \overline{uw}}{\partial r}$$

$$+ \overline{uw} \frac{\partial \overline{v^2}}{\partial r} + \frac{2}{r} \overline{uw}\, \overline{v^2} + 3\overline{uw} \frac{\partial \overline{w^2}}{\partial r}\}] \qquad \text{(E13f)}$$

Substituting (E13) into (E12) the diffusion terms becomes,

$$J^1_{11,1} = \frac{1}{r} \frac{\partial}{\partial r} [-r\, C_o \frac{\overline{q^2}}{\varepsilon} \{3(C_2+\tfrac{3}{5})\, \overline{u^2} \frac{\partial \overline{u^2}}{\partial r} + (C_2-\tfrac{2}{5})[\overline{u^2} \frac{\partial \overline{v^2}}{\partial r}$$

$$+ \overline{u^2} \frac{\partial \overline{w^2}}{\partial r} + 2\overline{uw} \frac{\partial \overline{uw}}{\partial r} + \frac{2}{r} (\overline{u^2 v^2} - \overline{v^2}^2)]\}]$$

$$+ \frac{2}{r} C_o \frac{\overline{q^2}}{\varepsilon} \{- \frac{3}{5} \overline{u^2} \frac{\partial \overline{u^2}}{\partial r} + \frac{4}{5} \overline{u^2} \frac{\partial \overline{v^2}}{\partial r} - \frac{1}{5} \overline{u^2} \frac{\partial \overline{w^2}}{\partial r}$$

$$- \frac{2}{5} \overline{uw} \frac{\partial \overline{uw}}{\partial r} + \frac{8}{5r} (\overline{u^2 v^2} - \overline{v^2}^2)\} \qquad \text{(E14a)}$$

$$J^1_{22,1} = r \frac{\partial}{\partial r} [-r\, C_o \frac{\overline{q^2}}{\varepsilon} \{3\, C_2\, \overline{u^2} \frac{\partial \overline{u^2}}{\partial r} + (C_2+1)\, \overline{u^2} \frac{\partial \overline{v^2}}{\partial r}$$

$$+ C_2\, \overline{u^2} \frac{\partial \overline{w^2}}{\partial r} + 2\, C_2\, \overline{uw} \frac{\partial \overline{uw}}{\partial r} + 2(C_2+1)\, \frac{1}{r} (\overline{u^2 v^2} - \overline{v^2}^2)\}]$$

$$- 2r\, C_o \frac{\overline{q^2}}{\varepsilon} \{- \frac{3}{5} \overline{u^2} \frac{\partial \overline{u^2}}{\partial r} + \frac{4}{5} \overline{u^2} \frac{\partial \overline{v^2}}{\partial r} - \frac{1}{5} \overline{u^2} \frac{\partial \overline{w^2}}{\partial r}$$

$$- \frac{2}{5} \overline{uw} \frac{\partial \overline{uw}}{\partial r} + \frac{8}{5} \frac{1}{r} (\overline{u^2 v^2} - \overline{v^2}^2)\} \qquad \text{(E14b)}$$

$$J^{1}_{33,1} = \frac{1}{r}\frac{\partial}{\partial r}[-r\ C_o \frac{\overline{q^2}}{\varepsilon} \{3\ C_2\ \overline{u^2}\ \frac{\partial \overline{u^2}}{\partial r} + C_2\ \overline{u^2}\ \frac{\partial \overline{v^2}}{\partial r}$$

$$+ (C_2+1)\ \overline{u^2}\ \frac{\partial \overline{w^2}}{\partial r} + 2(C_2+1)\ \overline{uw}\ \frac{\partial \overline{uw}}{\partial r}$$

$$+ C_2\ \frac{2}{r}\ (\overline{u^2v^2} - \overline{v^2}^2)\}] \qquad \text{(E14c)}$$

$$J^{1}_{13,1} = \frac{1}{r}\frac{\partial}{\partial r}[-r\ C_o \frac{\overline{q^2}}{\varepsilon} \{\frac{4}{5}\ \overline{uw}\ \frac{\partial \overline{u^2}}{\partial r} - \frac{1}{5}\ \overline{uw}\ \frac{\partial \overline{v^2}}{\partial r}$$

$$- \frac{3}{5}\ \overline{uw}\ \frac{\partial \overline{w^2}}{\partial r} + \frac{8}{5}\ \overline{u^2}\ \frac{\partial \overline{uw}}{\partial r} - \frac{2}{5}\ \frac{1}{r}\ \overline{uw}\ \overline{v^2}\}]$$

$$+ \frac{1}{r}\ C_o\ \frac{\overline{q^2}}{\varepsilon}\ \{-\ \frac{1}{5}\ \overline{uw}\ \frac{\partial \overline{u^2}}{\partial r} + \frac{4}{5}\ \overline{uw}\ \frac{\partial \overline{v^2}}{\partial r} - \frac{3}{5}\ \overline{uw}\ \frac{\partial \overline{w^2}}{\partial r}$$

$$- \frac{2}{5}\ \overline{u^2}\ \frac{\partial \overline{uw}}{\partial r} + \frac{8}{5}\ \frac{1}{r}\ \overline{uw}\ \overline{v^2}\} \qquad \text{(E14d)}$$

<u>The Rapid Terms</u>

The mean velocity graients $U_{p,}q$ transform to:

$$U_{p,q} = \frac{\partial U_p}{\partial x^q} - \Gamma^{m}_{pq}\ U_m \qquad \text{(E15)}$$

the non zero values of $U_{p,q}$ are given by:

$$U_{p,q} = \begin{vmatrix} \frac{\partial U}{\partial r} & 0 & 0 \\ 0 & rU & \\ \frac{\partial W}{\partial r} & 0 & \frac{\partial W}{\partial z} \end{vmatrix} \qquad \text{(E16)}$$

Hence with (E16) and (E5) the rapid terms can be evaluated and the results are:

$$\phi^{pq}_{11} = 4[\frac{1}{15} - cb_{11}]\overline{q^2}\ \frac{\partial U}{\partial r} + 4[-\ \frac{1}{30} + cb_{11} + cb_{22}]\overline{q^2}\ \frac{U}{r}$$

$$- \frac{4}{3}\ (1+5c)\ \overline{uw}\ \frac{\partial W}{\partial r} + 4[-\ \frac{1}{30} + cb_{11} + cb_{22}]\overline{q^2}\ \frac{\partial W}{\partial z} \qquad \text{(E17a)}$$

$$\phi_{22}^{pq} = 4[-\frac{1}{30} + cb_{33} + cb_{22}]\overline{q^2}\,\frac{\partial U}{\partial r} + 4[\frac{1}{15} - cb_{22}]\overline{q^2}\,\frac{U}{r}$$

$$+4c\,\overline{uw}\,\frac{\partial W}{\partial r} + 4[-\frac{1}{30} + cb_{33} + cb_{22}]\overline{q^2}\,\frac{\partial W}{\partial z} \qquad \text{(E17b)}$$

$$\phi_{33}^{pq} = 4[-\frac{1}{30} + cb_{11} + cb_{33}]\overline{q^2}\,\frac{\partial U}{\partial r} + 4[-\frac{1}{30} + cb_{33}$$

$$+ cb_{22}]\overline{q^2}\,\frac{U}{r} + \frac{4}{3}(1+2c)\,\overline{uw}\,\frac{\partial W}{\partial r} + 4[\frac{2}{30} - cb_{33}]\overline{q^2}\,\frac{\partial W}{\partial z} \qquad \text{(E17c)}$$

$$\phi_{12}^{pq} = -\,2c\,\overline{uw}\,\frac{U}{r} - 2c\,\overline{uw}\,\frac{U}{r} + [\frac{1}{10} + (1-c)\,\frac{\overline{u^2}}{\overline{q^2}} + c]\overline{q^2}\,\frac{\partial W}{\partial r}$$

$$c\,\overline{uw}\,\frac{\partial W}{\partial z} \qquad \text{(E17e)}$$

<u>Final Form of the Reynolds Stress Equations</u>

Now let us go back to the original notations as indicated in chapter 1. Hence (U, u) are the axial components (V,v) the radial component and (W,w) are a component of the mean and turbulence velocities respectively. Then the final form of the stress equations is given by;

<u>x-component</u>

$$U\frac{\partial\overline{u^2}}{\partial x} + V\frac{\partial\overline{u^2}}{\partial r} = 2\,\overline{uv}\,\frac{\partial U}{\partial r} - \underline{2\,\overline{u^2}\,\frac{\partial U}{\partial x}} + \frac{1}{r}\frac{\partial}{\partial r}[r\,C_0\,\frac{\overline{q^2}}{\varepsilon}\{3\,\overline{v^2}\,\frac{\partial\overline{v^2}}{\partial r}$$

$$+ C_2\,\overline{v^2}\,\frac{\partial\overline{v^2}}{\partial r} + (C_2+1)\,\overline{v^2}\,\frac{\partial\overline{u^2}}{\partial r} + 2(C_2+1)\,\overline{uv}\,\frac{\partial\overline{uv}}{\partial r}$$

$$+ \frac{2}{r}\,(\overline{v^2}\,\overline{w^2} - \overline{w^2}^2)\}] - C_1\varepsilon(\frac{\overline{u^2}}{\overline{q^2}} - \frac{1}{3}) + 4[-\frac{1}{30} + \underline{cb_{22}}$$

$$+ \underline{cb_{11}]\overline{q^2}}\,\frac{\partial V}{\partial r} + \underline{4[-\frac{1}{30} + cb_{11} + cb_{33}]\overline{q^2}\,\frac{V}{r}}$$

$$+ \frac{4}{3}\,(1+2c)\,\overline{uv}\,\frac{\partial U}{\partial r} + \underline{4[\frac{2}{30} - cb_{11}]\overline{q^2}\,\frac{\partial U}{\partial x}} - \frac{2}{3}\,\varepsilon \qquad \text{(E18a)}$$

r-component

$$U\frac{\partial\overline{v^2}}{\partial x} + V\frac{\partial\overline{v^2}}{\partial r} = \underline{2\,\overline{v^2}\,\frac{\partial V}{\partial r}} + \frac{1}{r}\frac{\partial}{\partial r}\left[r\, C_o\, \frac{\overline{q^2}}{\varepsilon}\left\{3\left(C_2+\frac{3}{5}\right)\overline{v^2}\,\frac{\partial\overline{v^2}}{\partial r}\right.\right.$$

$$+ \left(C_2-\frac{2}{3}\right)\left(\overline{v^2}\,\frac{\partial\overline{w^2}}{\partial r} + \overline{v^2}\,\frac{\partial\overline{u^2}}{\partial r} + 2\,\overline{uv}\,\frac{\partial\overline{uv}}{\partial r}\right.$$

$$\left.\left.\left. + \frac{2}{r}\left(\overline{v^2 w^2} - \overline{w^2}^2\right)\right)\right\}\right] - \frac{2}{r}\, C_o\, \frac{\overline{q^2}}{\varepsilon}\left\{-\frac{3}{5}\,\overline{v^2}\,\frac{\partial\overline{v^2}}{\partial r}\right.$$

$$+ \frac{4}{5}\,\overline{v^2}\,\frac{\partial\overline{w^2}}{\partial r} - \frac{1}{5}\,\overline{v^2}\,\frac{\partial\overline{u^2}}{\partial r} - \frac{2}{5}\,\overline{uv}\,\frac{\partial\overline{uv}}{\partial r} - \frac{8}{5r}$$

$$\left. - \frac{8}{5r}\left(\overline{v^2 w^2} - \overline{w^2}^2\right)\right\} - C_1\varepsilon\left(\frac{\overline{v^2}}{\overline{q^2}} - \frac{1}{3}\right) +$$

$$+ \underline{4\left[\frac{1}{15} - cb_{22}\right]\overline{q^2}\,\frac{\partial V}{\partial r}} + \underline{4\left[-\frac{1}{30} + cb_{22} + cb_{33}\right]\overline{q^2}\,\frac{V}{r}}$$

$$- \frac{4}{3}(1+5c)\,\overline{uv}\,\frac{\partial U}{\partial r} + \underline{4\left[-\frac{1}{30} + cb_{11} + cb_{22}\right]\overline{q^2}\,\frac{\partial U}{\partial x}} - \frac{2}{3}\,\varepsilon$$

(E18b)

θ-component

$$U\frac{\partial\overline{w^2}}{\partial x} + V\frac{\partial\overline{w^2}}{\partial r} = \underline{-\,2\,\overline{w^2}\,\frac{V}{r}} + \frac{1}{r}\frac{\partial}{\partial r}\left[r\, C_o\, \frac{\overline{q^2}}{\varepsilon}\left\{3C_2\,\overline{v^2}\,\frac{\partial\overline{v^2}}{\partial r}\right.\right.$$

$$+ (C_2+1)\overline{v^2}\,\frac{\partial\overline{w^2}}{\partial r} + C_2\,\overline{v^2}\,\frac{\partial\overline{u^2}}{\partial r} + 2C_2\,\overline{uv}\,\frac{\partial\overline{uv}}{\partial r}$$

$$\left.\left. + \frac{2}{r}\,(C_2+1)\left(\overline{v^2 w^2} - \overline{w^2}^2\right)\right\}\right] + \frac{2}{r}\, C_o\, \frac{\overline{q^2}}{\varepsilon}\left\{-\frac{3}{5}\,\overline{v^2}\,\frac{\partial\overline{v^2}}{\partial r}\right.$$

$$+ \frac{4}{5}\,\overline{v^2}\,\frac{\partial\overline{w^2}}{\partial r} - \frac{1}{5}\,\overline{v^2}\,\frac{\partial\overline{u^2}}{\partial r} - \frac{2}{5}\,\overline{uv}\,\frac{\partial\overline{uv}}{\partial r}$$

$$\left. + \frac{2}{5}\,\frac{1}{r}\left(\overline{v^2 w^2} - \overline{w^2}^2\right)\right\} - C_1\left(\frac{\overline{w^2}}{\overline{q^2}} - \frac{1}{3}\right)$$

$$+ \underline{4\left[-\frac{1}{30} + cb_{22} + cb_{33}\right]\overline{q^2}\,\frac{\partial V}{\partial r}} + \underline{4\left[\frac{1}{15} - cb_{33}\right]\overline{q^2}\,\frac{V}{r}}$$

$$+ 4\,c\overline{uv}\,\frac{\partial U}{\partial r} + \underline{4\left[-\frac{1}{30} + cb_{11} + cb_{33}\right]\overline{q^2}\,\frac{\partial U}{\partial x}} - \frac{2}{3} \qquad \text{(E18c)}$$

Shear Stress

$$U\frac{\partial\overline{uv}}{\partial x} + V\frac{\partial\overline{uv}}{\partial r} = \underline{-\overline{uv}}\,\frac{\partial V}{\partial r} - \overline{v^2}\,\frac{\partial U}{\partial r} - \underline{\overline{uv}}\,\frac{\partial U}{\partial x} + \frac{1}{r}\frac{\partial}{\partial r}\Big[r\, C_o\, \frac{\overline{q^2}}{\varepsilon}\Big\{\frac{4}{5}\,\overline{uv}\,\frac{\partial\overline{u^2}}{\partial r}$$

$$- \frac{1}{5}\,\overline{uv}\,\frac{\partial\overline{w^2}}{\partial r} - \frac{3}{5}\,\overline{uv}\,\frac{\partial\overline{u^2}}{\partial r} + \frac{8}{5}\,\overline{v^2}\,\frac{\partial\overline{uv}}{\partial r} - \frac{2}{5}\,\frac{1}{r}\,\overline{uv}\,\overline{w^2}\Big\}\Big]$$

$$- \frac{1}{r}\, C_o\, \frac{\overline{q^2}}{\varepsilon}\quad - \frac{1}{5}\,\overline{uv}\,\frac{\partial\overline{v^2}}{\partial r} + \frac{4}{5}\,\overline{uv}\,\frac{\partial\overline{w^2}}{\partial r} - \frac{3}{5}\,\overline{wv}\,\frac{\partial\overline{u^2}}{\partial r}$$

$$- \frac{2}{5}\,\overline{v^2}\,\frac{\partial\overline{uv}}{\partial r} + \frac{8}{5}\,\overline{uv}\,\frac{\overline{w^2}}{r}\quad - C_1\,\varepsilon\frac{\overline{uv}}{q^2} - \underline{2\, c\overline{uv}\,\frac{\partial V}{\partial r}} + \underline{4\,\overline{uv}\,\frac{V}{r}}$$

$$+ 2\Big[\frac{1}{10} + (1-c)\,\frac{\overline{v^2}}{q^2} - \frac{1}{3}\,(1+8c)\,\frac{\overline{u^2}}{q^2} + c\Big]\overline{q^2}\,\frac{\partial U}{\partial r} - \underline{2\,\alpha\overline{uv}\,\frac{\partial U}{\partial x}} \qquad \text{(E18d)}$$

$$- 4\, c\overline{uv}\,\frac{U}{x}$$

Where the underlined terms are of 2nd order based on order of magnitude analysis similar to that of Appendix D.

$$b_{11} = \frac{\overline{u^2}}{q^2} - \frac{1}{3} \qquad \text{(E19a)}$$

$$b_{22} = \frac{\overline{v^2}}{q^2} - \frac{1}{3} \qquad \text{(E19b)}$$

$$b_{33} = \frac{\overline{w^2}}{q^2} - \frac{1}{3} \qquad \text{(E19c)}$$

APPENDIX F

Numerical Scheme

The following numerical scheme solves any set of finite difference e equations simultaneously. This simple algorithm consists of the recursion relations f4, f8, and f9. In order to up-date E_j and F_j in the tridiagonalization, matrix inversion is required, at every node point j. Then by back substitution (equation F4) one can solve for the unknowns at the grid point j.

Consider a set of difference equations that results from a coupled system of differential equations, which can be, in general, written as follows.

$$\begin{aligned}
a_{11}^{1}x_{j-1}^{1} + b_{11}x_{j}^{1} + c_{11}x_{j+1}^{1} + a_{12}x_{j-1}^{2} + b_{12}x_{j}^{2} + \ldots + c_{1n}x_{j+1}^{n} &= d^{1} \\
a_{21}x_{j-1}^{1} + b_{21}x_{j}^{1} + c_{21}x_{j+1}^{1} + a_{22}x_{j-1}^{2} + b_{22}x_{j}^{2} + \ldots + c_{2n}x_{j+1}^{n} &= d^{2} \\
\vdots \qquad\qquad \vdots \qquad\qquad\qquad\qquad\qquad\qquad \vdots \quad & \quad \vdots \\
a_{n1}x_{j-1}^{1} + b_{n1}x_{j}^{1} + c_{n1}x_{j+1}^{1} + a_{n2}x_{j-1}^{2} + b_{n2}x_{j}^{2} + \ldots + c_{nn}x_{j+1}^{n} &= d^{n}
\end{aligned} \qquad \text{(F1)}$$

where the x's are the unknown values of the variables at the corresponding grid points (i.e., j-1, j and j+1 in Figure 3.2), the a's, b's and c's are the constant coefficients of the variables in questions and d's stand for the source terms in the system of equations. The above equation (E1) can be written in the following matrix form.

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x_{j-1}^{1} \\ x_{j-1}^{2} \\ \vdots \\ x_{j-1}^{n} \end{bmatrix} + \begin{bmatrix} b_{11} & b_{22} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nn} \end{bmatrix} \begin{bmatrix} x_{j}^{1} \\ x_{j}^{2} \\ \vdots \\ x_{j}^{n} \end{bmatrix}$$

$$+ \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2n} \\ \vdots & \vdots & & \vdots \\ c_{n1} & c_{n2} & \cdots & c_{nn} \end{bmatrix} \begin{bmatrix} x_{j+1}^{1} \\ x_{j+1}^{2} \\ \vdots \\ x_{j+1}^{n} \end{bmatrix} = \begin{bmatrix} d^{1} \\ d^{2} \\ \vdots \\ d^{n} \end{bmatrix} \qquad \text{(F2)}$$

Solution of the Set of Equations

Let us write the above system (F2) for the jth grid point as follows;

$$A_jX_{j-1} + B_jX_j + C_jX_{j+1} = D_j \tag{F3}$$

where A_j, B_j and C_j are the (nxn) coefficients matrices and D_j is the constant column vector in the above system where all are evaluated at the jth point. The X's stand for the unknowns at the corresponding grid points j-1, j and j+1. Now let us define,

$$X_j = F_j - E_jX_{j+1} \tag{F4}$$

where the matrices F_j and E_j will be determined later. If we replace j by j-1 in equation (F4) and substitute into equation (F3) we get,

$$A_j(F_{j-1} - E_{j-1}X_j) + B_jX_j + C_jX_{j+1} = D_j \tag{F5}$$

or by rearranging equation (F5) becomes

$$(B_j - A_jE_{j-1})\ X_j = D_j - A_jF_{j-1} - C_jX_{j+1} \tag{F6}$$

If we multiply equation (E6) by the inverse of the matrix $(B_j - A_jE_{j-1})$ the following equation for X_j will result

$$X_j = (B_j - A_jE_{j-1})^{-1}(D_j - A_jE_{j-1}) - (B_j - A_jE_{j-1})^{-1}C_jX_{j+1} \tag{F7}$$

Now if we compare equation (F7) with (F4) we find that;

$$E_j = (B_j - A_jE_{j-1})^{-1}\ C_j \tag{f8}$$

and

$$F_j = (B_j - A_jE_{j-1})^{-1}(D_j - A_jF_{j-1}) \tag{F9}$$

Equation (F9) can be evaluated only for j=2, ... J-1. Hence for j=1 (i.e. ξ=o) and j=J ($\xi=\xi_{max}$) we evaluate the system of equation (F2) using the appropriate boundary conditions.

i) Inner Boundary (ξ=0)

At the inner boundary we require that

$$A_1 = o \qquad \text{(F10)}$$

Hence it follows from equation (F3) that

$$B_1X_1 + C_1X_2 = D \qquad \text{(F11)}$$

and from equation (F8) and (F9) we have,

$$E_1 = B_1^{-1}\ C_1 \qquad \text{(F12)}$$

$$F_1 = B_1^{-1}\ D_1 \qquad \text{(F13)}$$

ii) Outer Boundary ($\xi=\xi_{max}$)

At the outer boundary we require that

$$C_J = 0 \qquad \text{(F14)}$$

then from (f3) we have,

$$A_JX_{J-1} + B_JX_J = D_J \qquad \text{(F15)}$$

and equations (F8) and (F9) become

$$E_J = 0 \qquad \text{(F16)}$$

$$F_J = (B_J - A_JE_{J-1})^{-1}(D_J - A_JF_{J-1}) \qquad \text{(F17)}$$

HIGHER ORDER CLOSURE MODEL FOR TURBULENT JETS

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